Gdańsk University
of Technology

Random Processes and Stochastic Control – Part 2 of 2

by

Maciej Niedźwiecki

Gdańsk University of Technology

Departament of Automatic Control

Gdańsk, Poland

Minimum-variance estimation
Estymacja minimalnowariancyjna

Problem formulation

Consider two correlated vector random variables $\mathbf{X}$ and $\mathbf{Y}$ and their outcomes $\mathbf{x} = \mathbf{X}(\xi)$ and $\mathbf{y} = \mathbf{Y}(\xi)$. Suppose that only $\mathbf{y}$ is known. Find such estimator of $\mathbf{x}$, based on $\mathbf{y}$ and further denoted by $\widehat{\mathbf{x}}(\mathbf{y})$, that minimizes the mean-square-error (MSE) criterion

$$J[\,\widehat{\mathbf{x}}\,] = \mathrm{E}\left[\|\,\mathbf{X} - \widehat{\mathbf{x}}\,\|^2\,|\mathbf{Y} = \mathbf{y}\right] \longrightarrow \min$$

mean-square-error / błąd średniokwadratowy

$$\widehat{\mathbf{x}}(\mathbf{y}) = ?$$

Theorem

The least-mean-squares (optimal mean-square-error) estimator of $\mathbf{X}$ given $\mathbf{Y}$ is the conditional expectation of $\mathbf{X}$ given $\mathbf{Y}$,i.e., $\widehat{\mathbf{X}} = \mathrm{E}[\mathbf{X}|\mathbf{Y}]$. The resulting estimate is

$$\widehat{\mathbf{x}}(\mathbf{y}) = \mathrm{E}[\mathbf{X}|\mathbf{Y} = \mathbf{y}] = \int_{\Omega_X} \mathbf{x}\,p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y})\,d\mathbf{x}$$

where Ω_X denotes the support (domain) of the random variable $\mathbf{X}$.

conditional mean estimator / estymator średniej warunkowej

Minimum-variance estimation
Estymacja minimalnowariancyjna

Proof

Note that

$$
\begin{aligned}
J[\,\widehat{\mathbf{x}}\,] &= \int_{\Omega_X} (\mathbf{x}-\widehat{\mathbf{x}})^{\mathrm{T}}(\mathbf{x}-\widehat{\mathbf{x}}) p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) d\mathbf{x} \\
&= \int_{\Omega_X} \mathbf{x}^{\mathrm{T}}\mathbf{x} p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) d\mathbf{x} \\
&\quad - 2\widehat{\mathbf{x}}^{\mathrm{T}} \int_{\Omega_X} \mathbf{x} p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) d\mathbf{x} + \widehat{\mathbf{x}}^{\mathrm{T}}\widehat{\mathbf{x}}
\end{aligned}
$$

Requiring

$$
\nabla_{\widehat{\mathbf{x}}} = \frac{\partial J[\,\widehat{\mathbf{x}}\,]}{\partial \widehat{\mathbf{x}}} = \mathbf{0}
$$

one arrives at

$$
\widehat{\mathbf{x}} = \int_{\Omega_X} \mathbf{x} p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) d\mathbf{x} = \mathrm{E}[\mathbf{X}|\mathbf{Y}=\mathbf{y}]
$$

∎

Important property of the optimal estimator
Ważna własność estymatora optymalnego

The conditional mean estimator is unbiased, i.e.,

$$\mathrm{E}[\mathbf{X} - \mathrm{E}[\mathbf{X}|\mathbf{Y}]] = \mathbf{0}$$

which follows directly from the fact that $\mathrm{E}[\mathbf{X}] = \mathbf{m}_{\mathbf{X}}$ and

$$\mathrm{E}_{\mathbf{Y}}[\mathrm{E}_{\mathbf{X}}[\mathbf{X}|\mathbf{Y}]] = \int_{\Omega_Y}\int_{\Omega_X} \mathbf{x} p_{\mathbf{X}|\mathbf{Y}}(\mathbf{x}|\mathbf{y}) p_{\mathbf{Y}}(\mathbf{y}) d\mathbf{x} d\mathbf{y}$$
$$= \int_{\Omega_Y}\int_{\Omega_X} \mathbf{x} p_{\mathbf{X}\mathbf{Y}}(\mathbf{x},\mathbf{y}) d\mathbf{x} d\mathbf{y} = \int_{\Omega_X} \mathbf{x} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = \mathbf{m}_{\mathbf{X}}$$

Moreover, for any function of $\mathbf{Y}$, say $g(\mathbf{Y})$, it holds that

$$\mathrm{E}\{\, [\mathbf{X} - \mathrm{E}[\mathbf{X}|\mathbf{Y}]]\, g(\mathbf{Y})\} = \mathbf{0}$$

which means that the estimation error is orthogonal to (actually, also uncorrelated with) any transformation of the data $\mathbf{y}$, i.e., no matter how we modify the data $\mathbf{y}$ the estimation accuracy cannot be further improved.

Evaluation of the posterior density

Wyznaczanie gęstości rozkładu a posteriori

Bayes formula / wzór Bayesa

Assuming that $p_Y(\mathbf{y}) \neq 0$

$$p_{X|Y}(\mathbf{x}|\mathbf{y}) = \frac{p_{XY}(\mathbf{x},\mathbf{y})}{p_Y(\mathbf{y})} = \frac{p_{Y|X}(\mathbf{y}|\mathbf{x})p_X(\mathbf{x})}{\int_{\Omega_\mathbf{x}} p_{Y|X}(\mathbf{y}|\mathbf{x})p_X(\mathbf{x})d\mathbf{x}}$$

$p_X(\mathbf{x})$ – prior distribution (rozkład *a priori*), which reflects our knowledge about $\mathbf{x}$ prior to measuring $\mathbf{y}$

$p_{Y|X}(\mathbf{y}|\mathbf{x})$ – conditional distribution which quantifies dependence between the estimated value $\mathbf{x}$ and the measurement $\mathbf{y}$

$p_{X|Y}(\mathbf{x}|\mathbf{y})$ – posterior distribution (rozkład *a posteriori*), which reflects our knowledge about $\mathbf{x}$ after measuring $\mathbf{y}$

Thomas Bayes (1702 - 1761)

Scalar example
Przykład skalarny

Suppose that

$$y = x + w$$

where y, x and w are realizations of scalar random variables Y, X and W, respectively. Find the least-mean-squares estimate of x given y if it is known that the variables X and W are independent and

$$X \sim \mathcal{N}(x_0, \sigma_0^2), \quad W \sim \mathcal{N}(0, \sigma_W^2)$$

<u>Solution</u>

$$p_{Y|X}(y|x) = p_W(y-x) = \frac{1}{\sqrt{2\pi\sigma_W^2}} \exp\left\{-\frac{(y-x)^2}{2\sigma_W^2}\right\}$$

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(x-x_0)^2}{2\sigma_0^2}\right\}$$

$$\begin{aligned} p_{X|Y}(x|y) &= c\, p_{Y|X}(y|x) p_X(x) \\ &= \frac{c}{2\pi\sqrt{\sigma_W^2\sigma_0^2}} \exp\left\{-\frac{(y-x)^2}{2\sigma_W^2} - \frac{(x-x_0)^2}{2\sigma_0^2}\right\} \end{aligned}$$

One can show that

$$p_Y(y) = \frac{1}{c} = \int_{-\infty}^{\infty} p_{Y|X}(y|x) p_X(x) dx$$
$$= \frac{1}{\sqrt{2\pi(\sigma_W^2 + \sigma_0^2)}} \exp\left\{ -\frac{y^2}{2(\sigma_W^2 + \sigma_0^2)} \right\}$$

and

$$\frac{(y-x)^2}{2\sigma_W^2} + \frac{(x-x_0)^2}{2\sigma_0^2} - \frac{y^2}{2(\sigma_W^2+\sigma_0^2)}$$
$$= \frac{\left(x - \frac{\sigma_W^2 x_0 + \sigma_0^2 y}{\sigma_W^2 + \sigma_0^2} \right)^2}{2 \, \frac{\sigma_W^2 \sigma_0^2}{\sigma_W^2 + \sigma_0^2}}$$

which leads to

$$p_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\sigma_{X|Y}^2}} \exp\left\{ -\frac{(x - m_{X|Y})^2}{2\sigma_{X|Y}^2} \right\}$$

where

$$m_{X|Y} = \frac{\sigma_W^2 x_0 + \sigma_0^2 y}{\sigma_W^2 + \sigma_0^2}, \quad \sigma_{X|Y}^2 = \frac{\sigma_W^2 \sigma_0^2}{\sigma_W^2 + \sigma_0^2}$$

Therefore

$$\widehat{x} = \mathrm{E}[X|Y=y] = m_{X|Y} = \frac{\sigma_W^2 x_0 + \sigma_0^2 y}{\sigma_W^2 + \sigma_0^2} = \frac{\frac{x_0}{\sigma_0^2} + \frac{y}{\sigma_W^2}}{\frac{1}{\sigma_0^2} + \frac{1}{\sigma_W^2}}$$

$$\mathrm{var}[\widehat{x}] = \mathrm{E}[(X - \widehat{x})^2 | Y = y] = \sigma_{X|Y}^2 = \frac{\sigma_W^2 \sigma_0^2}{\sigma_W^2 + \sigma_0^2}$$

Important relationships for vector Gaussian variables

Ważne zależności dla wektorowych zmiennych gaussowskich

Consider two vector random variables $\mathbf{X}_{k\times 1}$ and $\mathbf{Y}_{n\times 1}$ which are jointly normally distributed

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{m}_X \\ \mathbf{m}_Y \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_X & \mathbf{\Sigma}_{XY} \\ \mathbf{\Sigma}_{YX} & \mathbf{\Sigma}_Y \end{bmatrix} \right)$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_X & \mathbf{\Sigma}_{XY} \\ \mathbf{\Sigma}_{YX} & \mathbf{\Sigma}_Y \end{bmatrix} > 0$$

Then the conditional distribution of $\mathbf{X}$ given $\mathbf{Y}$ is also normal

$$\begin{aligned} p_{X|Y}(\mathbf{x}|\mathbf{y}) &\sim \mathcal{N}(\mathbf{m}_{X|Y}, \mathbf{\Sigma}_{X|Y}) \\ \mathbf{m}_{X|Y} &= \mathbf{m}_X + \mathbf{\Sigma}_{XY}\mathbf{\Sigma}_Y^{-1}(\mathbf{y} - \mathbf{m}_Y) \\ \mathbf{\Sigma}_{X|Y} &= \mathbf{\Sigma}_X - \mathbf{\Sigma}_{XY}\mathbf{\Sigma}_Y^{-1}\mathbf{\Sigma}_{YX} \end{aligned}$$

Vector example
Przykład wektorowy

Suppose that

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{w}$$

where $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{w}$ are realizations of scalar random variables $\mathbf{Y}$, $\mathbf{X}$ and $\mathbf{W}$, respectively. Find the least-mean-squares estimate of $\mathbf{x}$ given $\mathbf{y}$ if it is known that the variables $\mathbf{X}$ and $\mathbf{W}$ are independent and

$$\mathbf{X} \sim \mathcal{N}(\mathbf{x}_0, \mathbf{\Sigma}_0), \quad \mathbf{W} \sim \mathcal{N}(0, \mathbf{W})$$

Solution

$$\widehat{\mathbf{x}} = \mathbf{m}_X + \mathbf{\Sigma}_{XY}\mathbf{\Sigma}_Y^{-1}(\mathbf{y} - \mathbf{m}_Y)$$

Since

$$\mathbf{m}_X = \mathbf{x}_0, \quad \mathbf{m}_Y = \mathbf{A}\mathbf{m}_X = \mathbf{A}\mathbf{x}_0$$

one obtains

$$\mathbf{y} - \mathbf{m}_Y = \mathbf{A}(\mathbf{x} - \mathbf{m}_X) + \mathbf{w}$$

Note that

$$\begin{aligned}
\mathbf{\Sigma}_X &= \mathrm{E}[(\mathbf{X} - \mathbf{m}_X)(\mathbf{X} - \mathbf{m}_X)^{\mathrm{T}}] = \mathbf{\Sigma}_0 \\
\mathbf{\Sigma}_{XY} &= \mathrm{E}[(\mathbf{X} - \mathbf{m}_X)(\mathbf{Y} - \mathbf{m}_Y)^{\mathrm{T}}] = \mathbf{\Sigma}_0\mathbf{A}^{\mathrm{T}} \\
\mathbf{\Sigma}_Y &= \mathrm{E}[(\mathbf{Y} - \mathbf{m}_Y)(\mathbf{Y} - \mathbf{m}_Y)^{\mathrm{T}}] = \mathbf{A}\mathbf{\Sigma}_0\mathbf{A}^{\mathrm{T}} + \mathbf{W}
\end{aligned}$$

Therefore (provided that $\mathbf{A}\mathbf{\Sigma}_0\mathbf{A}^{\mathrm{T}} + \mathbf{W} > 0$)

$$\widehat{\mathbf{x}} = \mathbf{x}_0 + \mathbf{\Sigma}_0\mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{\Sigma}_0\mathbf{A}^{\mathrm{T}} + \mathbf{W})^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x}_0)$$

Signal prediction, filtration and smoothing
Predykcja, filtracja i wygładzanie sygnałów

Assume that the m-dimensional signal $\mathbf{s}(t)$ admits the following state space description

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{v}(t)$$
$$\mathbf{s}(t) = \mathbf{C}\mathbf{x}(t)$$
$$\mathbf{y}(t) = \mathbf{s}(t) + \mathbf{w}(t)$$

where $\mathbf{x}(t)$ denotes the n-dimensional state vector, $\mathbf{v}(t)$ denotes the n-dimensional driving noise vector, $\mathbf{y}(t)$ denotes the m-dimensional measurement vector, $\mathbf{w}(t)$ denotes the m-dimensional measurement noise vector, and $\mathbf{A}_{n\times n}$, $\mathbf{C}_{m\times n}$ are known matrices.

Find out the estimate of $\mathbf{s}(t)$ based on the available data record $\mathcal{Y}(T) = \{\mathbf{y}(1), \ldots, \mathbf{y}(T)\}$

$$\mathcal{Y}(T) \longrightarrow \widehat{\mathbf{s}}(t|T), \widehat{\mathbf{x}}(t|T)$$

$$\widehat{\mathbf{s}}(t|T) = \mathbf{C}\widehat{\mathbf{x}}(t|T)$$

Important cases:

$T < t$	-	prediction / predykcja
$T = t$	-	filtration / filtracja
$T > t$	-	smoothing / wygładzanie

State space model of an autoregressive signal

Model stanowy sygnału autoregresyjnego

Consider an autoregressive signal governed by

$$s(t) = \sum_{i=1}^{r} a_i s(t-i) + n(t)$$

where $a_1, \ldots, a_r$ denote autoregressive coefficients and $\{n(t)\}$ denotes zero-mean white noise.

It is easy to check that $s(t)$ has the following state space representation (one of infinitely many)

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{v}(t)$$
$$\mathbf{s}(t) = \mathbf{C}\mathbf{x}(t)$$

where

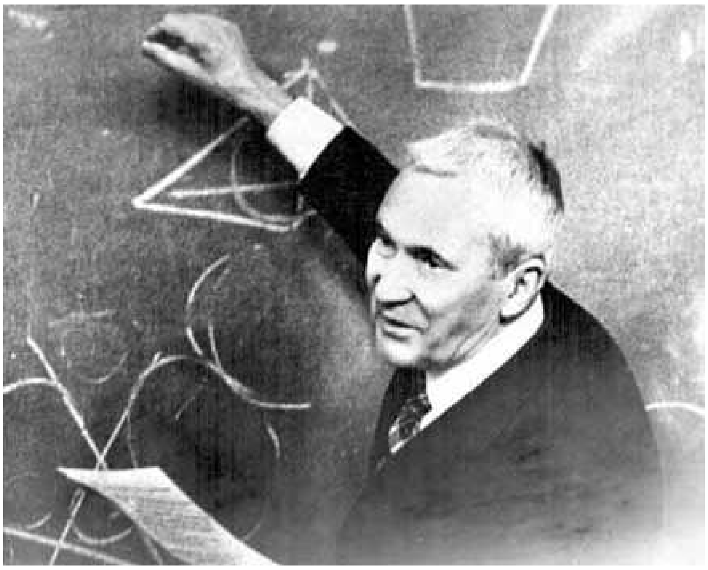
$$\mathbf{x}(t) = [s(t), \ldots, s(t-r+1)]^{\mathrm{T}}$$

$$\mathbf{A} = \begin{bmatrix} a_1 & \ldots & a_{r-1} & a_r \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad \mathbf{v}(t) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} n(t+1)$$

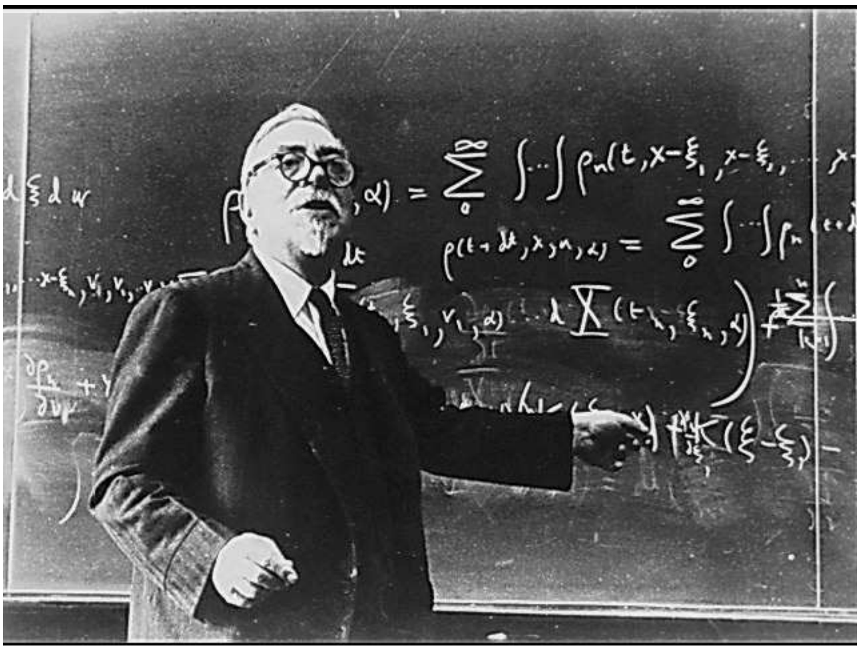
$$\mathbf{C} = [1, 0, \ldots, 0]$$

Hall of Fame

Galeria Sław



Andrei Kolmogorov (1903 - 1987)
"Interpolation and extrapolation," Bull. Acad. Sci. URSS, pp. 3-14, 1941.



Norbert Wiener (1894 - 1964)
Extrapolation, Interpolation and Smoothing of Stationary Time Series, MIT Press, 1949.

Hall of Fame
Galeria Sław

Rudolf Kalman (1930)
"A new approach to linear filering and prediction problems," Journal of Basic Engineering, Transactions of ASME, vol. 82, pp. 35-45, 1960.

Kalman filter/predictor – assumptions
Filtr/predyktor Kalmana – założenia

We will assume that the noisy signal admits the following state space description

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{v}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t)$$

where $\{\mathbf{v}(t)\}$ and $\{\mathbf{w}(t)\}$ are zero-mean and mutually independent white Gaussian noise sequences with known covariance matrices $\mathbf{V}_{n\times n}$ and $\mathbf{W}_{m\times m}$

$$\mathbf{v}(t) \sim \mathcal{N}(0, \mathbf{V})$$
$$\mathbf{w}(t) \sim \mathcal{N}(0, \mathbf{W})$$

The initial state $\mathbf{x}(0)$ is Gaussian and independent of $\{\mathbf{v}(t)\}$ and $\{\mathbf{w}(t)\}$

$$\mathbf{x}(0) \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0)$$

Kalman filter/predictor – terminology
Filtr/predyktor Kalmana – terminologia

Under Gaussian assumption it holds that

$$p(\mathbf{x}(t)|\mathcal{Y}(t-1)) = \mathcal{N}(\widehat{\mathbf{x}}(t|t-1), \mathbf{P}(t|t-1))$$
$$p(\mathbf{x}(t)|\mathcal{Y}(t)) = \mathcal{N}(\widehat{\mathbf{x}}(t|t), \mathbf{P}(t|t))$$

where

$$\widehat{\mathbf{x}}(t|t-1) = \mathrm{E}[\mathbf{x}(t)|\mathcal{Y}(t-1)]$$

predicted (*a priori*) state estimate
predykcyjna (*a priori*) ocena stanu

$$\widehat{\mathbf{x}}(t|t) = \mathrm{E}[\mathbf{x}(t)|\mathcal{Y}(t)]$$

filtered (*a posteriori*) state estimate
filtracyjna (*a posteriori*) ocena stanu

$\mathbf{P}(t|t-1) = \mathrm{cov}[\widehat{\mathbf{x}}(t|t-1)]$ and $\mathbf{P}(t|t) = \mathrm{cov}[\widehat{\mathbf{x}}(t|t)]$ denote the *a priori* and *a posteriori* covariance matrices of the corresponding estimation errors

Three types of recursive algorithms:

$\widehat{\mathbf{x}}(t|t-1) \longrightarrow \widehat{\mathbf{x}}(t+1|t)$ Kalman predictor

$\widehat{\mathbf{x}}(t|t) \longrightarrow \widehat{\mathbf{x}}(t+1|t+1)$ Kalman filter

$\widehat{\mathbf{x}}(t|t-1) \longrightarrow \widehat{\mathbf{x}}(t|t)$

$\longrightarrow \widehat{\mathbf{x}}(t+1|t)$ Kalman filter/predictor

Kalman filter/predictor – basic steps
Filtr/predyktor Kalmana – podstawowe kroki

Kalman filter/predictor is a recursive estimation algorithm which evaluates minimum-variance state estimates:

1. Evaluate the one-step-ahead state prediction $\widehat{\mathbf{x}}(t|t-1)$ (*a priori* state estimate).

2. Evaluate the one-step-ahead measurement prediction $\mathbf{C}\widehat{\mathbf{x}}(t|t-1)$.

3. Take a new measurement $\mathbf{y}(t)$.

4. Evaluate the one-step-ahead measurement prediction error $\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \mathbf{C}\widehat{\mathbf{x}}(t|t-1)$

5. Evaluate the corrected state estimate $\widehat{\mathbf{x}}(t|t)$ that incorporates information contained in a new measurement (*a posteriori* state estimate)

a posteriori state estimate $=$ *a priori* state estimate
$+$ gain $\times$ (measurement prediction error)

Kalman filter/predictor – algorithm
Filtr/predyktor Kalmana – algorytm

initial conditions / warunki początkowe

$$\widehat{\mathbf{x}}(0|0) = \mathbf{x}_0$$
$$\mathbf{P}(0|0) = \mathbf{P}_0$$

time update / aktualizacja czasu

$$\widehat{\mathbf{x}}(t+1|t) = \mathbf{A}\widehat{\mathbf{x}}(t|t)$$
$$\mathbf{P}(t+1|t) = \mathbf{A}\mathbf{P}(t|t)\mathbf{A}^{\mathrm{T}} + \mathbf{V}$$

measurement update / aktualizacja pomiarów

$$\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \mathbf{C}\widehat{\mathbf{x}}(t|t-1)$$
$$\mathbf{S}(t) = \mathbf{C}\mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}} + \mathbf{W}$$
$$\mathbf{K}(t) = \mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}\mathbf{S}^{-1}(t)$$
$$\widehat{\mathbf{x}}(t|t) = \widehat{\mathbf{x}}(t|t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t)$$
$$\mathbf{P}(t|t) = \mathbf{P}(t|t-1) - \mathbf{K}(t)\mathbf{S}(t)\mathbf{K}^{\mathrm{T}}(t)$$

$\mathbf{K}(t)$ – Kalman gain / macierz wzmocnień filtru Kalmana

The notion of innovation
Pojęcie innowacji

The one-step-ahead measurement prediction error $\boldsymbol{\varepsilon}(t)$ is called innovation (*innowacja*). Innovation means "new information". The name stems from the observation that when $\boldsymbol{\varepsilon}(t) = 0$ the estimation update takes the form

$$\widehat{\mathbf{x}}(t|t) = \widehat{\mathbf{x}}(t|t-1)$$

In such a case the measurement $\mathbf{y}(t)$ does not bring any new information that could be incorporated in the process of estimation of $\mathbf{x}(t)$. The only piece of information that is new is contained in $\boldsymbol{\varepsilon}(t)$.

Important properties of the innovation process:

- $\boldsymbol{\varepsilon}(t)$ is uncorrelated with past measurements, namely

$$\mathrm{E}\{\boldsymbol{\varepsilon}(t) g[\mathcal{Y}(t-1)]\} = 0$$

  where $g[\mathcal{Y}(t-1)]$ denotes any function of $\mathcal{Y}(t-1)$.

- $\{\boldsymbol{\varepsilon}(t)\}$ is a sequence of Gaussian, zero-mean mutually uncorrelated random variables with covariance matrix $\mathbf{S}(t)$:

$$\boldsymbol{\varepsilon}(t) \sim \mathcal{N}(0, \mathbf{S}(t)) \ , \quad \mathrm{E}[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}(\tau)] = 0, \forall t \neq \tau$$

Observation 1

The properties of the innovation process can be used to eliminate false (unreliable) measurements. The following consistency test is often used

$$\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\mathbf{S}^{-1}(t)\boldsymbol{\varepsilon}(t) \leq \eta_0 \ ?$$

where η_0 is the decision threshold determined empirically, e.g. $\eta_0 = 9$.

When the condition above is not fulfilled, one is recommended to discard (how?) the measurement $\mathbf{y}(t)$.

Observation 2

It holds that

$$\mathbf{P}(t|t) \leq \mathbf{P}(t|t-1), \quad \forall t$$

which means that a posteriori state estimates are generally more accurate than the a priori estimate (since they incorporate an additional piece of information contained in the measurement $\mathbf{y}(t)$).

Observation 3

The values of the matrices $\mathbf{P}(t+1|t)$, $\mathbf{P}(t|t)$ and $\mathbf{K}(t)$ do not depend on measurements, and therefore they can be computed prior to running Kalman algorithm and saved in the computer memory.

Observation 4

When state space equations describe a control system

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{v}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t)$$

where $\mathbf{u}(t)$ denotes the measurable input (control) signal, only one Kalman recursion should be modified

$$\widehat{\mathbf{x}}(t+1|t) = \mathbf{A}\widehat{\mathbf{x}}(t|t) + \mathbf{B}\mathbf{u}(t)$$

Observation 5

Consider a time-varying system governed by

$$\mathbf{x}(t+1) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{v}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{w}(t)$$

$$\mathrm{cov}[\mathbf{v}(t)] = \mathbf{V}(t), \quad \mathrm{cov}[\mathbf{w}(t)] = \mathbf{W}(t)$$

Then the structure of Kalman filter/predictor does not change. The only difference with respect to the time-invariant case is that the constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{V}$ and $\mathbf{W}$ are replaced with their time-varying counterparts.

Kalman filter/predictor for nonstationary signals/systems

Filtr/predyktor Kalmana dla niestacjonarnych sygnałów/układów

$$\widehat{\mathbf{x}}(t+1|t) = \mathbf{A}(t)\widehat{\mathbf{x}}(t|t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{P}(t+1|t) = \mathbf{A}(t)\mathbf{P}(t|t)\mathbf{A}^{\mathrm{T}}(t) + \mathbf{V}(t)$$

$$\boldsymbol{\varepsilon}(t) = \mathbf{y}(t) - \mathbf{C}(t)\widehat{\mathbf{x}}(t|t-1)$$

$$\mathbf{S}(t) = \mathbf{C}(t)\mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}(t) + \mathbf{W}(t)$$

$$\mathbf{K}(t) = \mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}(t)\mathbf{S}^{-1}(t)$$

$$\widehat{\mathbf{x}}(t|t) = \widehat{\mathbf{x}}(t|t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t)$$

$$\mathbf{P}(t|t) = \mathbf{P}(t|t-1) - \mathbf{K}(t)\mathbf{S}(t)\mathbf{K}^{\mathrm{T}}(t)$$

Stationary Kalman filter – Wiener filter
Stacjonarny filtr Kalmana – filtr Wienera

When the analyzed process is stationary and asymptotically stable, i.e.,

$$|\lambda_i(\mathbf{A})| < 1, \quad i = 1, \ldots, n$$

Kalman filter/predictor is asymptotically stationary and asymptotically stable

$$\lim_{t \to \infty} \mathbf{P}(t|t-1) = \mathbf{P}_\infty, \quad \lim_{t \to \infty} \mathbf{K}(t) = \mathbf{K}_\infty$$

The steady state matrix $\mathbf{P}_\infty > 0$ is the positive definite solution of the following algebraic Riccati equation

$$\mathbf{P}_\infty = \mathbf{A}\left[\mathbf{P}_\infty - \mathbf{P}_\infty\mathbf{C}^\mathrm{T}(\mathbf{C}\mathbf{P}_\infty\mathbf{C}^\mathrm{T} + \mathbf{W})^{-1}\mathbf{C}\mathbf{P}_\infty\right]\mathbf{A}^\mathrm{T} + \mathbf{V}$$

and the steady state Kalmain gain can be obtained from

$$\mathbf{K}_\infty = \mathbf{P}_\infty\mathbf{C}^\mathrm{T}(\mathbf{C}\mathbf{P}_\infty\mathbf{C}^\mathrm{T} + \mathbf{W})^{-1}$$

The steady state predictor takes the form

$$\begin{aligned} \widehat{\mathbf{x}}(t+1|t) = {} & (\mathbf{A} - \mathbf{A}\mathbf{K}_\infty\mathbf{C})\widehat{\mathbf{x}}(t|t-1) \\ & + \mathbf{A}\mathbf{K}_\infty\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t) \end{aligned}$$

and is always stable: $|\lambda_i(\mathbf{A} - \mathbf{A}\mathbf{K}_\infty\mathbf{C})| < 1, i = 1, \ldots, n.$

Kalman filter as a state observer

Filtr Kalmana jako obserwator stanu

State observer for the system governed by

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

has the following form

$$\begin{aligned}\widehat{\mathbf{x}}(t+1) &= \mathbf{A}\widehat{\mathbf{x}}(t) + \mathbf{L}[\mathbf{y}(t) - \mathbf{C}\widehat{\mathbf{x}}(t)] + \mathbf{B}\mathbf{u}(t) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\widehat{\mathbf{x}}(t) + \mathbf{L}\mathbf{y}(t) + \mathbf{B}\mathbf{u}(t)\end{aligned}$$

Since the state estimation error $\widetilde{\mathbf{x}}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$ is governed by

$$\widetilde{\mathbf{x}}(t+1) = (\mathbf{A} - \mathbf{L}\mathbf{C})\widetilde{\mathbf{x}}(t)$$

the estimated state converges to the true state iff $|\lambda_i(\mathbf{A} - \mathbf{L}\mathbf{C})| < 1$, $i = 1, \ldots, n$.

If the system is observable, one can choose the observer gain $\mathbf{L}$ so as to place the eigenvalues of the matrix $(\mathbf{A} - \mathbf{L}\mathbf{C})$ in *any* prescribed positions.

In the presence of driving noise and measurement noise the error equation takes the form

$$\widetilde{\mathbf{x}}(t+1) = (\mathbf{A} - \mathbf{L}\mathbf{C})\widetilde{\mathbf{x}}(t) - \mathbf{L}\mathbf{w}(t) + \mathbf{v}(t)$$

$\mathbf{L} = \mathbf{A}\mathbf{K}_\infty$ is the gain that *minimizes* the mean square state estimation error.

Example (academic)
Przykład (akademicki)

Design Kalman filter for estimation of the state in the following system

$$
\begin{aligned}
x(t+1) &= x(t) \\
y(t) &= x(t) + w(t)
\end{aligned}
$$

$$w(t) \sim \mathcal{N}(0, \sigma_w^2), \quad x(0) \sim \mathcal{N}(x_0, \sigma_0^2)$$

In the case considered Kalman recursions take the following form

$$
\begin{aligned}
\widehat{x}(t+1|t) &= \widehat{x}(t|t) \\
p(t+1|t) &= p(t|t) \\
k(t) &= \frac{p(t|t-1)}{p(t|t-1) + \sigma_w^2} \\
\varepsilon(t) &= y(t) - \widehat{x}(t-1|t-1) \\
\widehat{x}(t|t) &= \widehat{x}(t-1|t-1) + k(t)\varepsilon(t) \\
p(t|t) &= \frac{p(t|t-1)\sigma_w^2}{p(t|t-1) + \sigma_w^2}
\end{aligned}
$$

with initial conditions: $\widehat{x}(0|0) = x_0$ and $p(0|0) = \sigma_0^2$.

Note that

$$p(1|1) = \frac{\sigma_w^2}{1 + \sigma_w^2/\sigma_0^2}\,, \quad p(2|2) = \frac{\sigma_w^2}{2 + \sigma_w^2/\sigma_0^2}\,, \quad \ldots$$

and, more generally,

$$p(t|t) = \frac{\sigma_w^2}{t + \sigma_w^2/\sigma_0^2}\,, \quad k(t|t) = \frac{1}{t + \sigma_w^2/\sigma_0^2}$$

which leads to

$$\begin{aligned}\widehat{x}(t|t) &= \widehat{x}(t-1|t-1) \\ &\qquad + \frac{1}{t + \sigma_w^2/\sigma_0^2}\,[y(t) - \widehat{x}(t-1|t-1)], \quad \widehat{x}(0|0) = x_0\end{aligned}$$

We will compare this algorithm with an "intuitive" solution

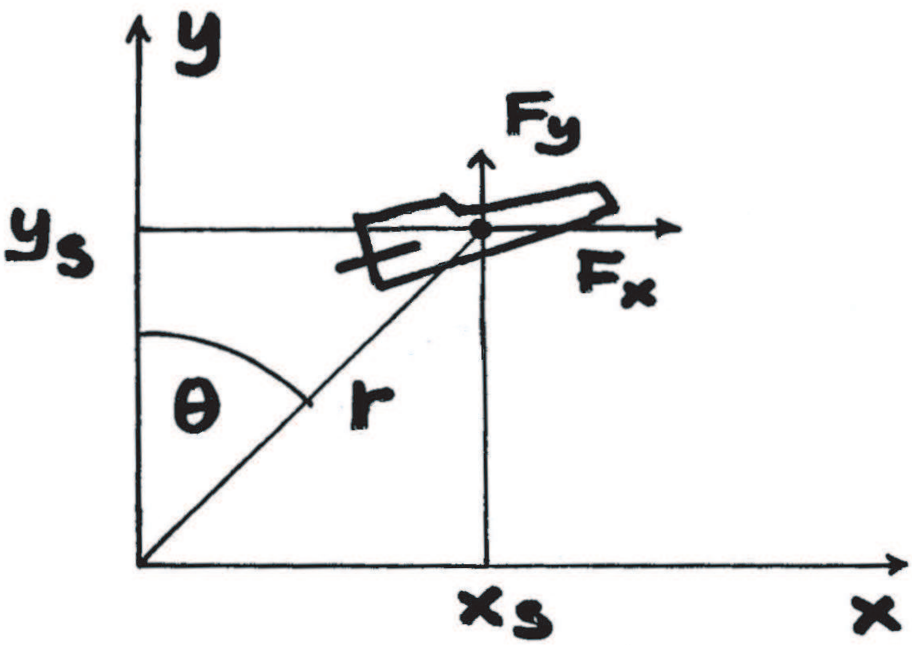
$$\bar{x}(t) = \frac{1}{t}\sum_{i=1}^{t} y(i)$$

which can be also put down in a recursive form

$$\begin{aligned}\bar{x}(t) &= \frac{(t-1)\bar{x}(t-1) + y(t)}{t} \\ &= \bar{x}(t-1) + \frac{1}{t}\,[y(t) - \bar{x}(t-1)], \quad \bar{x}(0) = 0\end{aligned}$$

Example (non-academic) – target tracking
Przykład (nieakademicki) – śledzenie obiektów

Estimate position (x_s, y_s) and speed $(\dot{x}_s, \dot{y}_s)$ of the aircraft, based on radar measurements of the distance r and azimuth θ (see the figure below). For simplicity restrict analysis to two dimensions: x and y.



Assuming that the forces F_x, F_y and the mass m of the aircraft are known, differential equations describing its dynamics can be written down in the form

$$\ddot{x}_s = \frac{F_x}{m} = u_x$$

$$\ddot{y}_s = \frac{F_y}{m} = u_y$$

Target tracking

Let $\mathbf{x} = [x_s, \dot{x}_s, y_s, \dot{y}_s]^{\mathrm{T}}$ and $\mathbf{u} = [u_x, u_y]^{\mathrm{T}}$.

The continuous-time state space model of the plane has the form

$$\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c\mathbf{u}$$

where

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}$$

Assuming that the control signal $\mathbf{u}$ is constant during each sampling interval of length T_s, one arrives at the following discrete-time state space model

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

where

$$\mathbf{A} = e^{\mathbf{A}_c T_s} = \begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \int_0^{T_s} e^{\mathbf{A}_c\tau}\mathbf{B}_c d\tau = \begin{bmatrix} T_s^2/2 & 0 \\ T_s & 0 \\ 0 & T_s^2/2 \\ 0 & T_s \end{bmatrix}$$

Target tracking – state equation

Since the forces F_x and F_y are not known, we will model $\{\mathbf{u}(t)\}$ as a random process with zero mean and covariance matrix

$$\mathbf{U} = \mathrm{cov}[\mathbf{u}(t)] = \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & \sigma_u^2 \end{bmatrix}, \quad \sigma_u^2 = \frac{(\Delta u_{\max})^2}{T_s}$$

where $\Delta u_{\max}$ denotes the maximum admissible acceleration change in the interval of length T_s.

In this way, one arrives at the following state equation

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{v}(t)$$

where $\{\mathbf{v}(t)\}$ denotes the so-called *maneuver noise*

$$\mathbf{v}(t) = \mathbf{B}\mathbf{u}(t)$$

and

$$\mathbf{V} = \mathrm{cov}[\mathbf{v}(t)] = \mathbf{B}\mathrm{cov}[\mathbf{u}(t)]\mathbf{B}^{\mathrm{T}} = \mathbf{B}\mathbf{U}\mathbf{B}^{\mathrm{T}} =$$

$$= \sigma_u^2 \begin{bmatrix} T_s^4/4 & T_s^3/2 & 0 & 0 \\ T_s^3/2 & T_s^2 & 0 & 0 \\ 0 & 0 & T_s^4/4 & T_s^3/2 \\ 0 & 0 & T_s^3/2 & T_s^2 \end{bmatrix}$$

Target tracking – output equation

Note that

$$\begin{bmatrix} x_s(t) \\ y_s(t) \end{bmatrix} = \begin{bmatrix} r_0(t)\sin\theta_0(t) \\ r_0(t)\cos\theta_0(t) \end{bmatrix} = g[r_0(t), \theta_0(t)]$$

where $r_0(t)$ and $\theta_0(t)$ denote the true distance and the true azimuth, respectively.

It holds that

$$r(t) = r_0(t) + \Delta r(t)$$
$$\theta(t) = \theta_0(t) + \Delta\theta(t)$$

$$\mathbf{Z} = \operatorname{cov}\left\{\begin{bmatrix} \Delta r(t) \\ \Delta\theta(t) \end{bmatrix}\right\} = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}$$

The quantity

$$\mathbf{y}(t) = \begin{bmatrix} r(t)\sin\theta(t) \\ r(t)\cos\theta(t) \end{bmatrix} = g[r(t), \theta(t)]$$

will be further called *pseudoobservation*.

Using Taylor series expansion, one arrives at

$$g(r,\theta) \cong g(r_0,\theta_0) - \left[\frac{\partial g}{\partial r} \;\; \frac{\partial g}{\partial \theta}\right] \begin{bmatrix} \Delta r \\ \Delta\theta \end{bmatrix}$$

Target tracking – output equation

This leads to the following output (measurement) equation

$$\mathbf{y}(t) \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \mathbf{w}(t)$$

where

$$\mathbf{w}(t) = -\mathbf{\Sigma}(t) \begin{bmatrix} \Delta r(t) \\ \Delta\theta(t) \end{bmatrix}$$

$$\mathbf{\Sigma}(t) = \begin{bmatrix} \sin\theta(t) & r(t)\cos\theta(t) \\ \cos\theta(t) & -r(t)\sin\theta(t) \end{bmatrix}$$

$$\mathbf{W}(t) = \mathrm{cov}[\mathbf{w}(t)] = \mathbf{\Sigma}(t)\mathrm{cov}\left\{\begin{bmatrix} \Delta r(t) \\ \Delta\theta(t) \end{bmatrix}\right\}\mathbf{\Sigma}^{\mathrm{T}}(t)$$
$$= \begin{bmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{bmatrix}$$

$$w_{11}(t) = \sigma_r^2[\sin\theta(t)]^2 + r^2(t)\sigma_\theta^2[\cos\theta(t)]^2$$
$$w_{22}(t) = \sigma_r^2[\cos\theta(t)]^2 + r^2(t)\sigma_\theta^2[\sin\theta(t)]^2$$
$$w_{12}(t) = [\sigma_r^2 - r^2(t)\sigma_\theta^2]\sin\theta(t)\cos\theta(t)$$
$$= w_{21}(t)$$

Extended Kalman filter (EKF)
Rozszerzony filtr Kalmana

Consider a nonlinear dynamic system governed by

$$\mathbf{x}(t+1) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] + \mathbf{v}(t)$$
$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t)] + \mathbf{w}(t)$$

Suppose we already have the estimates $\widehat{\mathbf{x}}(t|t-1)$ and $\widehat{\mathbf{x}}(t|t)$. Assuming that the estimation errors $||\mathbf{x}(t) - \widehat{\mathbf{x}}(t|t-1)||$ and $||\mathbf{x}(t) - \widehat{\mathbf{x}}(t|t)||$ are small, at the instant t one can use the following linearizations:

$$\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] \cong \mathbf{f}[\widehat{\mathbf{x}}(t|t), \mathbf{u}(t)] + \mathbf{A}(t|t)[\mathbf{x}(t) - \widehat{\mathbf{x}}(t|t)]$$
$$\mathbf{h}[\mathbf{x}(t)] \cong \mathbf{h}[\widehat{\mathbf{x}}(t|t-1)] + \mathbf{C}(t|t-1)[\mathbf{x}(t) - \widehat{\mathbf{x}}(t|t-1)]$$

where the matrices $\mathbf{A}(t|t)_{n\times n}$ and $\mathbf{C}(t|t-1)_{m\times n}$ are given by

$$\mathbf{A}(t|t) = \left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\substack{\mathbf{u}=\mathbf{u}(t)\\ \mathbf{x}=\widehat{\mathbf{x}}(t|t)}}$$
$$\mathbf{C}(t|t-1) = \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\mathbf{x}=\widehat{\mathbf{x}}(t|t-1)}$$

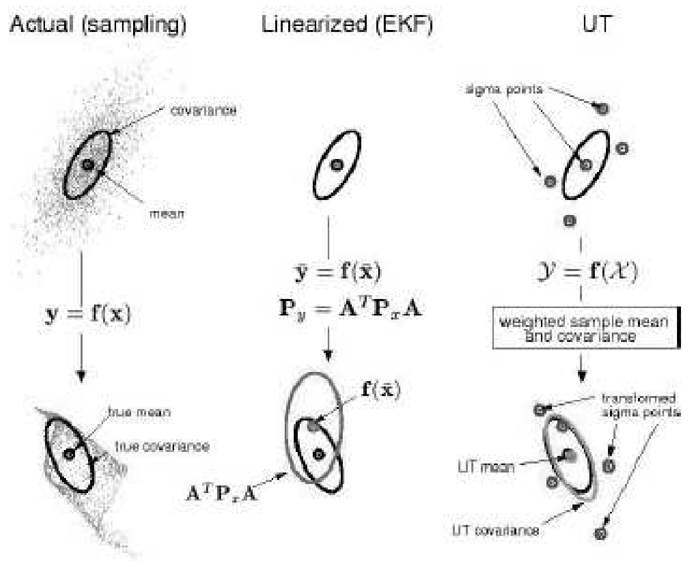
Extended Kalman filter (EKF)

extended Kalman filter = Kalman filter designed for a linearized system

$$\begin{aligned}
\widehat{\mathbf{x}}(t+1|t) &= \mathbf{f}[\widehat{\mathbf{x}}(t|t), \mathbf{u}(t)] \\
\mathbf{P}(t+1|t) &= \mathbf{A}(t|t)\mathbf{P}(t|t)\mathbf{A}^{\mathrm{T}}(t|t) + \mathbf{V}(t) \\
\boldsymbol{\varepsilon}(t) &= \mathbf{y}(t) - \mathbf{h}[\widehat{\mathbf{x}}(t|t-1)] \\
\mathbf{S}(t) &= \mathbf{C}(t|t-1)\mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}(t|t-1) + \mathbf{W}(t) \\
\mathbf{K}(t) &= \mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}(t|t-1)\mathbf{S}^{-1}(t) \\
\widehat{\mathbf{x}}(t|t) &= \widehat{\mathbf{x}}(t|t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t) \\
\mathbf{P}(t|t) &= \mathbf{P}(t|t-1) - \mathbf{K}(t)\mathbf{S}(t)\mathbf{K}^{\mathrm{T}}(t)
\end{aligned}$$

Does EKF stink ?

unscented transform



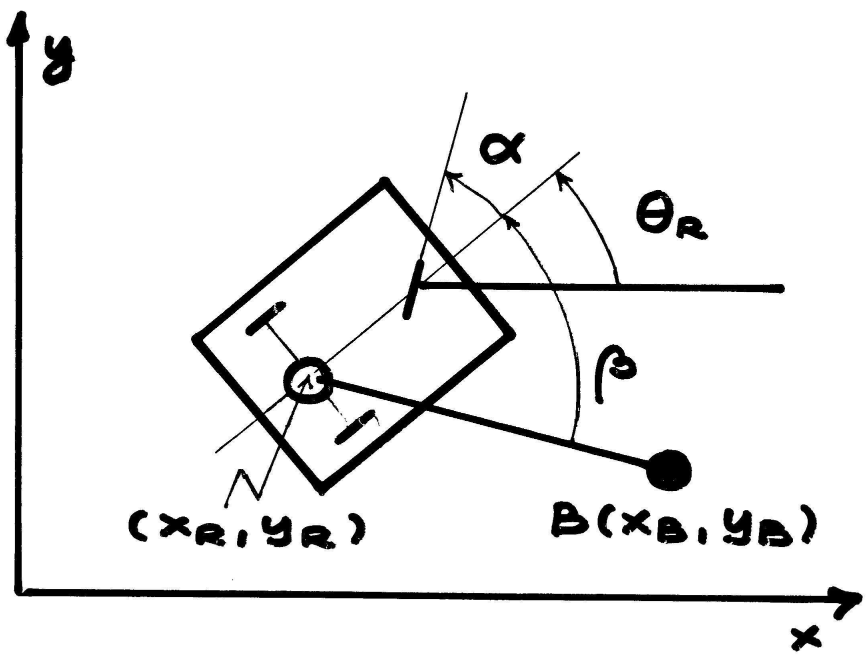
unscented Kalman filter (UKF)
bezwonny filtr Kalmana

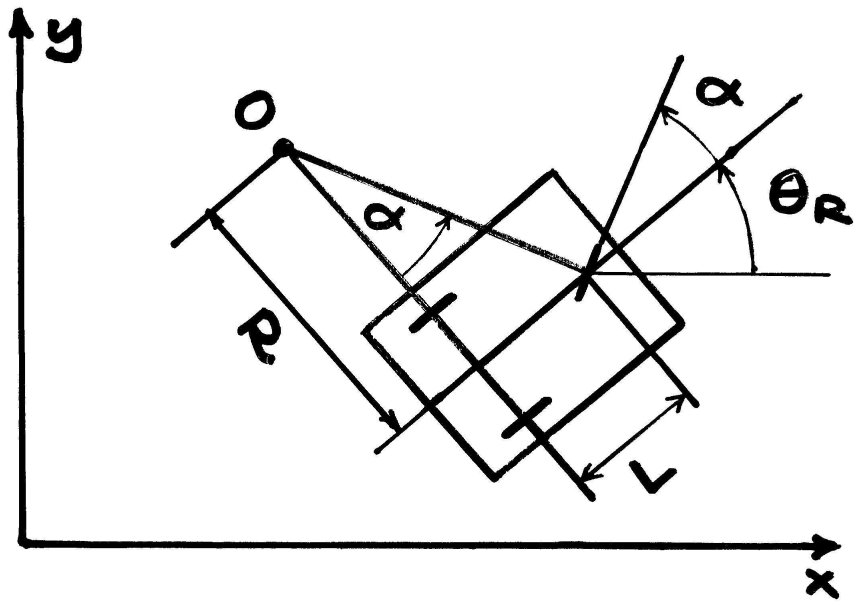
Example (non-academic) – sensor fusion

Przykład (nieakademicki) – łączenie informacji pochodzących z różnych czujników

Estimate position $(x_{\mathrm{R}}, y_{\mathrm{R}})$ and orientation θ_{R} of a 3-wheel mobile robot, based on measurements of:

- distance $d(t)$ covered by the robot during the last sampling interval (dead-reckoning based on measuring rotation of the robot wheels)
- angle $\alpha(t)$ of the steering wheel
- angle $\beta(t)$ to a beacon B with known coordinates $(x_{\mathrm{B}}, y_{\mathrm{B}})$





$$\dot{x}_{\mathrm{R}} = v \cos\theta_{\mathrm{R}}$$

$$\dot{y}_{\mathrm{R}} = v \sin\theta_{\mathrm{R}}$$

$$\dot{\theta}_{\mathrm{R}} = \frac{v}{R} = \frac{v}{L}\,\mathrm{tg}\alpha$$

Assuming that the speed of the robot $v = d/T_s$ and the steering angle α are constant in each sampling interval, one arrives at the following set of discrete-time equations

$$x_{\mathrm{R}}(t+1) = x_{\mathrm{R}}(t) + \frac{L}{\mathrm{tg}\alpha(t)}\left[\sin\theta_{\mathrm{R}}(t+1) - \sin\theta_{\mathrm{R}}(t)\right]$$

$$y_{\mathrm{R}}(t+1) = y_{\mathrm{R}}(t) - \frac{L}{\mathrm{tg}\alpha(t)}\left[\cos\theta_{\mathrm{R}}(t+1) - \cos\theta_{\mathrm{R}}(t)\right]$$

$$\theta_{\mathrm{R}}(t+1) = \theta_{\mathrm{R}}(t) + \frac{d(t)}{L}\,\mathrm{tg}\alpha(t)$$

$$\beta(t) = \mathrm{arctg}\,\frac{y_{\mathrm{B}} - y_{\mathrm{R}}(t)}{x_{\mathrm{B}} - x_{\mathrm{R}}(t)} + \theta_{\mathrm{R}}(t)$$

State space description / Opis w przestrzeni stanów

Using the notation

$$\mathbf{x}(t) = \left[\begin{array}{c} x_{\mathrm{R}}(t) \\ y_{\mathrm{R}}(t) \\ \theta_{\mathrm{R}}(t) \end{array} \right], \quad \mathbf{u}(t) = \left[\begin{array}{c} d(t) \\ \alpha(t) \end{array} \right], \quad y(t) = \beta(t)$$

one can rewrite system equations in the form

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] \\ y(t) &= h[\mathbf{x}(t)] \end{aligned}$$

where

$$\mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] =$$

$$\left[\begin{array}{c} x_{\mathrm{R}}(t) + \frac{L}{\mathrm{tg}\alpha(t)} \left[\sin\left(\theta_{\mathrm{R}}(t) + \frac{d(t)}{L}\,\mathrm{tg}\alpha(t) \right) - \sin\theta_{\mathrm{R}}(t) \right] \\ y_{\mathrm{R}}(t) - \frac{L}{\mathrm{tg}\alpha(t)} \left[\sin\left(\theta_{\mathrm{R}}(t) + \frac{d(t)}{L}\,\mathrm{tg}\alpha(t) \right) - \sin\theta_{\mathrm{R}}(t) \right] \\ \theta_{\mathrm{R}}(t) + \frac{d(t)}{L}\,\mathrm{tg}\alpha(t) \end{array} \right]$$

$$h[\mathbf{x}(t)] = \mathrm{arctg}\, \frac{y_{\mathrm{B}} - y_{\mathrm{R}}(t)}{x_{\mathrm{B}} - x_{\mathrm{R}}(t)} + \theta_{\mathrm{R}}(t)$$

Measurement errors / Błędy pomiarowe

Note that

$$\mathbf{u}(t) = \mathbf{u}_0(t) + \Delta\mathbf{u}(t)$$
$$\beta(t) = \beta_0(t) + \Delta\beta(t)$$

$$\mathrm{cov}[\Delta\mathbf{u}(t)] = \begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_\alpha^2 \end{bmatrix} = \mathbf{U}$$
$$\mathrm{var}[\Delta\beta(t)] = \sigma_\beta^2$$

where $\Delta\mathbf{u}(t)$ and $\Delta\beta(t)$ denote input and output measurement errors, respectively.

Since it holds that

$$\mathbf{f}[\mathbf{x}(t), \mathbf{u}_0(t)] \cong \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] + \mathbf{D}(t|t)[\mathbf{u}_0(t) - \mathbf{u}(t)]$$

where

$$\mathbf{D}(t|t) = \left.\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}^{\mathrm{T}}}\right|_{\substack{\mathbf{u}=\mathbf{u}(t) \\ \mathbf{x}=\widehat{\mathbf{x}}(t|t)}}$$

one arrives at the following description that accounts for measurement errors

$$\mathbf{x}(t+1) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] + \mathbf{v}(t)$$
$$y(t) = h[\mathbf{x}(t)] + w(t)$$

where $\mathbf{v}(t) = -\mathbf{D}(t|t)\Delta\mathbf{u}(t)$ and $w(t) = \Delta\beta(t)$.

System matrices / Macierze systemowe

$$
\begin{aligned}
\mathbf{V}(t) &= \operatorname{cov}[\mathbf{v}(t)] = \mathrm{E}[\mathbf{v}(t)\mathbf{v}^{\mathrm{T}}(t)] \\
&= \mathrm{E}\left[\mathbf{D}(t|t)\Delta\mathbf{u}(t)\Delta\mathbf{u}^{\mathrm{T}}(t)\mathbf{D}^{\mathrm{T}}(t|t)\right] \\
&= \mathbf{D}(t|t)\mathrm{E}\left[\Delta\mathbf{u}(t)\Delta\mathbf{u}^{\mathrm{T}}(t)\right]\mathbf{D}^{\mathrm{T}}(t|t) \\
&= \mathbf{D}(t|t)\mathbf{U}\mathbf{D}^{\mathrm{T}}(t|t) \\
\sigma_w^2 &= \sigma_\beta^2
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C}(t|t-1) &= \left.\frac{\partial h(\mathbf{x})}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\mathbf{x}=\widehat{\mathbf{x}}(t|t-1)} \\
&= \left[-\frac{y_{\mathrm{B}} - \widehat{y}_{\mathrm{R}}(t|t-1)}{r^2(t)}, \frac{x_{\mathrm{B}} - \widehat{x}_{\mathrm{R}}(t|t-1)}{r^2(t)}, 1\right]
\end{aligned}
$$

where

$$
r^2(t) = [x_{\mathrm{B}} - \widehat{x}_{\mathrm{R}}(t|t-1)]^2 + [y_{\mathrm{B}} - \widehat{y}_{\mathrm{R}}(t|t-1)]^2
$$

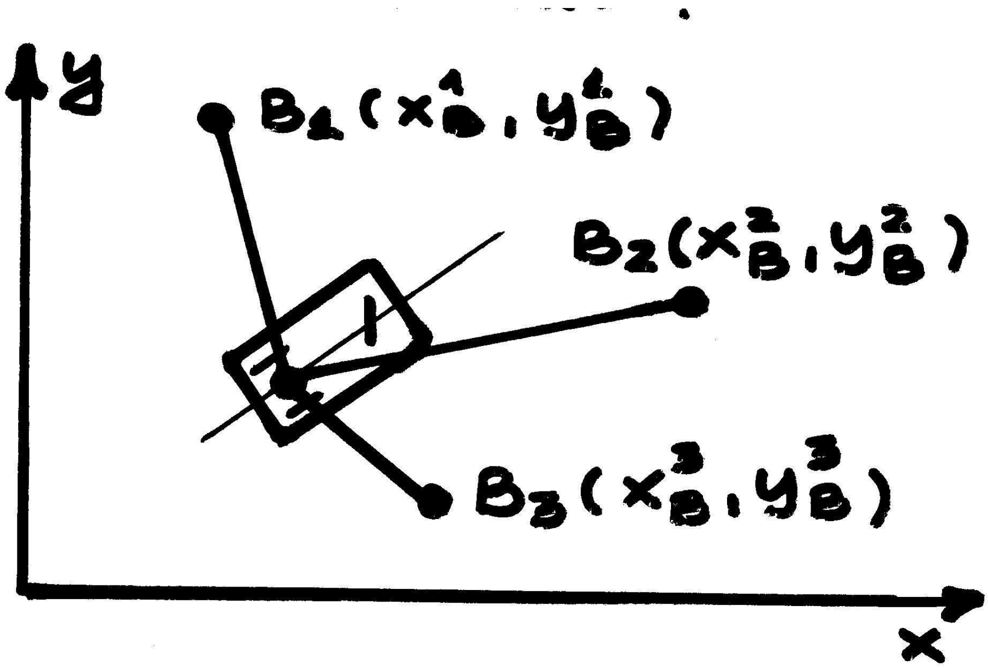
In a similar way one can evaluate

$$
\mathbf{A}(t|t) = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}^{\mathrm{T}}}\right|_{\substack{\mathbf{u}=\mathbf{u}(t)\\ \mathbf{x}=\widehat{\mathbf{x}}(t|t)}}
$$

$$
\mathbf{D}(t|t) = \left.\frac{\partial \mathbf{f}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}^{\mathrm{T}}}\right|_{\substack{\mathbf{u}=\mathbf{u}(t)\\ \mathbf{x}=\widehat{\mathbf{x}}(t|t)}}
$$

Incorporation of several beacons

Wykorzystanie kilku znaczników



$$\mathbf{y}(t) = \mathbf{h}[\mathbf{x}(t)] = \begin{bmatrix} h_1[\mathbf{x}(t)] \\ h_2[\mathbf{x}(t)] \\ h_3[\mathbf{x}(t)] \end{bmatrix}$$

How to eliminate outliers ?

Jak wyeliminować błędy grube?

$$\boldsymbol{\varepsilon}^{\mathrm{T}}(t)\mathbf{S}^{-1}(t)\boldsymbol{\varepsilon}(t) \leq \eta_0 \ ?$$

Numerical stability of Kalman filter
Stabilność numeryczna filtru Kalmana

Since $\mathbf{P}(t|t-1)$ and $\mathbf{P}(t|t)$ are covariance matrices of the corresponding state estimates, they must be symmetric and positive definite. Both properties may be lost due to finite precision of calculations.

Remedy 1 (symmetry / symetria)

$$\mathbf{P}(t|t-1) = [\mathbf{P}(t|t-1) + \mathbf{P}^{\mathrm{T}}(t|t-1)]/2$$
$$\mathbf{P}(t|t) = [\mathbf{P}(t|t) + \mathbf{P}^{\mathrm{T}}(t|t)]/2$$

Remedy 2 (positive definiteness / dodatnia określoność)

The main source of numerical ill-conditioning is due to the presence of subtraction in the *a posteriori* covariance matrix update

$$\mathbf{P}(t|t) = \mathbf{P}(t|t-1) - \mathbf{K}(t)\mathbf{S}(t)\mathbf{K}^{\mathrm{T}}(t)$$

The following modification, due to Josephs, is free of this drawback

$$\mathbf{P}(t|t) = [\mathbf{I}-\mathbf{K}(t)\mathbf{C}]\mathbf{P}(t|t-1)[\mathbf{I}-\mathbf{K}(t)\mathbf{C}]^{\mathrm{T}}+\mathbf{K}(t)\mathbf{W}\mathbf{K}^{\mathrm{T}}(t)$$

but does not guarantee that the matrix $\mathbf{P}(t|t)$ will be always positive definite.

Numerical stability of Kalman filter
Stabilność numeryczna filtru Kalmana

Remedy 2 (symmetry & positive definiteness)

Well-definiteness of the matrices $\mathbf{P}(t|t)$, $\mathbf{P}(t|t-1)$ and $\mathbf{S}(t)$ can be ensured if estimation is carried out using square roots of all covariance matrices involved in computation of $\mathbf{x}(t|t-1)$ and $\mathbf{x}(t|t)$. Let

$$\widetilde{\mathbf{P}}(t|t-1)_{n\times n} = \mathbf{P}^{1/2}(t|t-1)$$
$$\widetilde{\mathbf{P}}(t|t)_{n\times n} = \mathbf{P}^{1/2}(t|t)$$
$$\widetilde{\mathbf{S}}(t)_{m\times m} = \mathbf{S}^{1/2}(t)$$
$$\widetilde{\mathbf{V}}_{n\times n} = \mathbf{V}^{1/2}$$
$$\widetilde{\mathbf{W}}_{m\times m} = \mathbf{W}^{1/2}$$

which means that

$$\widetilde{\mathbf{P}}(t|t-1)\widetilde{\mathbf{P}}^{\mathrm{T}}(t|t-1) = \mathbf{P}(t|t-1)$$
$$\widetilde{\mathbf{P}}(t|t)\widetilde{\mathbf{P}}^{\mathrm{T}}(t|t) = \mathbf{P}(t|t)$$
$$\widetilde{\mathbf{S}}(t)\widetilde{\mathbf{S}}^{\mathrm{T}}(t) = \mathbf{S}(t)$$
$$\widetilde{\mathbf{V}}\widetilde{\mathbf{V}}^{\mathrm{T}} = \mathbf{V}$$
$$\widetilde{\mathbf{W}}\widetilde{\mathbf{W}}^{\mathrm{T}} = \mathbf{W}$$

square root of a matrix / pierwiastek kwadratowy macierzy

Square root Kalman filter/predictor
Pierwiastkowy filtr/predyktor Kalmana

time update / aktualizacja czasu

$$\widehat{\mathbf{x}}(t+1|t) = \mathbf{A}\widehat{\mathbf{x}}(t|t)$$

$$\mathbf{P}(t+1|t) = \mathbf{A}\mathbf{P}(t|t)\mathbf{A}^{\mathrm{T}} + \mathbf{V}$$

Find such an orthogonal $2n \times 2n$ matrix $\mathbf{Q}_p$

$$\mathbf{Q}_p\mathbf{Q}_p^{\mathrm{T}} = \mathbf{I}$$

which converts the matrix

$$\begin{bmatrix} \mathbf{A}\widetilde{\mathbf{P}}(t|t) & \widetilde{\mathbf{V}} \end{bmatrix}_{n \times 2n}$$

into the lower block triangular form, namely

$$\begin{bmatrix} \mathbf{A}\widetilde{\mathbf{P}}(t|t) & \widetilde{\mathbf{V}} \end{bmatrix} \mathbf{Q}_p = \begin{bmatrix} \widetilde{\mathbf{P}}(t+1|t) & \mathbf{O} \end{bmatrix}$$

Square root Kalman filter/predictor
Pierwiastkowy filtr/predyktor Kalmana

measurement update / aktualizacja pomiarów

$$\widehat{\mathbf{x}}(t|t) = \widehat{\mathbf{x}}(t|t-1) + \mathbf{K}(t)\boldsymbol{\varepsilon}(t)$$
$$\mathbf{P}(t|t) = \mathbf{P}(t|t-1) - \mathbf{K}(t)\mathbf{S}(t)\mathbf{K}^{\mathrm{T}}(t)$$

Find such an orthogonal $(n+m) \times (n+m)$ matrix $\mathbf{Q}_f$

$$\mathbf{Q}_f\mathbf{Q}_f^{\mathrm{T}} = \mathbf{I}$$

which converts the matrix

$$\begin{bmatrix} \widetilde{\mathbf{W}} & \mathbf{C}\widetilde{\mathbf{P}}(t|t-1) \\ \mathbf{O} & \widetilde{\mathbf{P}}(t|t-1) \end{bmatrix}_{(n+m)\times(n+m)}$$

into the lower block triangular form, namely

$$\begin{bmatrix} \widetilde{\mathbf{W}} & \mathbf{C}\widetilde{\mathbf{P}}(t|t-1) \\ \mathbf{O} & \widetilde{\mathbf{P}}(t|t-1) \end{bmatrix} \mathbf{Q}_f = \begin{bmatrix} \widetilde{\mathbf{S}}(t) & \mathbf{O} \\ \mathbf{K}(t)\widetilde{\mathbf{S}}(t) & \widetilde{\mathbf{P}}(t|t) \end{bmatrix}$$

$$\widehat{\mathbf{x}}(t|t) = \widehat{\mathbf{x}}(t|t-1) + \left[\mathbf{K}(t)\widetilde{\mathbf{S}}(t)\right] \widetilde{\mathbf{S}}^{-1}(t)\boldsymbol{\varepsilon}(t)$$

Approaches to matrix triangularization
Podejścia do triangularyzacji macierzy

The most often used methods:

- Givens rotations / obroty Givensa
- Hausholder transformations / przekształcenia Hauseholdera
- Gram-Schmidt transformations / przekształcenia Grama-Schmidta

Givens rotations

An interesting class of square root algorithms can be obtained by applying circular plane (Givens) rotations. Denote by $T_{ij}(A) = [t_{ij}], j > i$, the $k \times k$ circular plane rotation matrix which can be used to *annihilate* (reduce to zero) the (i, j) element of an arbitrary $k \times l,\ l \geq k$, matrix $A = [a_{ij}]$. Let

$$c = \cos\alpha = \frac{1}{\sqrt{1+\rho^2}}$$

$$s = \sin\alpha = \frac{\rho}{\sqrt{1+\rho^2}}$$

$$\alpha = \mathrm{arctg}\rho\ , \quad \rho = \frac{a_{ij}}{a_{ii}}$$

Givens rotations
Obroty Givensa

$$T_{ij}(A) = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & c & & & & -s & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & s & & & & c & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}$$

$$t_{ii} = t_{jj} = c\ , \quad t_{ij} = -\,t_{ji} = s$$

It is straightforward to check that

$$T_{ij}T_{ij}^T = I_k, \quad [AT_{ij}]_{ij} = 0$$

Note that

1. Since T_{ij} differs from the identity matrix only in its $(i,i),(i,j),(j,i)$ and (j,j) components, only two columns of A (i and j) are modified when rotation is executed.

2. If the superdiagonal elements of the ith column of A are zero ($a_{1i} = \ldots = a_{i-1,i} = 0$), rotation preserves all zero elements of the jth column located above a_{ij} ($a_{1j}, \ldots, a_{i-1,j}$).

Givens rotations
Obroty Givensa

One can use both properties for sequential triangularization purposes, selecting the rotation matrix in the form (row sweep)

$$T = \prod_{i=1}^{k} \left(\prod_{j=i+1}^{l} T_{ij} \right) = T_r$$

or in the form (column sweep)

$$T = \prod_{j=2}^{l} \left(\prod_{i=1}^{\min(j-1,k)} T_{ij} \right) = T_c$$

Such transforms are equivalent to performing a sequence of Givens rotations, each of which annihilates a particular element of the upper triangle of the transformed matrix. Using this approach one gets (schematically)

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} T = \begin{bmatrix} x & 0 & 0 & 0 \\ x & x & 0 & 0 \\ x & x & x & 0 \end{bmatrix}$$

In the case of 3×4 matrices

$$T_r = T_{12} T_{13} T_{14} T_{23} T_{24} T_{34} \,, \quad T_c = T_{12} T_{13} T_{23} T_{14} T_{24} T_{34}$$

Givens rotations
Obroty Givensa

Caution is needed when interpreting the relationships listed above. Strictly speaking, the operation

$$AT_{i_1j_1}T_{i_2j_2}\dots T_{i_mj_m} = B$$

should be written in the form

$$\begin{aligned}
B_1 &= AT_{i_1j_1}(A), \\
B_2 &= B_1T_{i_2j_2}(B_1), \\
&\vdots \\
B_m &= B_{m-1}T_{i_mj_m}(B_{m-1}), \\
B &= B_m,
\end{aligned}$$

i.e. the sine and cosine parameters of the orthogonal matrix $T_{i_lj_l}$, $1 \le l \le m$, cannot be evaluated before the first $l-1$ transformations are actually performed.

Kalman smoothing
Wygładzanie Kalmana

$$\widehat{\mathbf{x}}(t|T) = \mathrm{E}[\mathbf{x}(t)|\Omega(T)] \; , T > t$$

fixed-lag smoothing

wygładzanie ze stałym opóźnieniem

$$\widehat{\mathbf{x}}(t-\tau|t) \; , \; t = \tau+1, \tau+2, \ldots$$

fixed-interval smoothing

wygładzanie ze stałym przedziałem

$$\widehat{\mathbf{x}}(t|N) \; , \; t = 1, \ldots, N$$

fixed-point smoothing

wygładzanie stałopunktowe

$$\widehat{\mathbf{x}}(t_0|t) \; , \; t = t_0, t_0+1, \ldots$$

Fixed-interval smoothing
Wygładzanie ze stałym przedziałem

Mayne-Fraser smoothing formula

The smoothed estimate $\widehat{\mathbf{x}}(t|N)$ and its covariance matrix $\mathbf{P}(t|N)$ can be obtained by combining the estimates yielded by the forward Kalman filter and backward Kalman predictor, or equivalently, by combining the results provided by the forward Kalman predictor and backward Kalman filter. The resulting smoothing formula is known as the Mayne-Fraser two-filter algorithm

$$\begin{aligned}
\widehat{\mathbf{x}}(t|N) &= \mathbf{P}(t|N)\left[\, \mathbf{P}^{-1}(t|t)\widehat{\mathbf{x}}(t|t) \right. \\
&\qquad \left. +\mathbf{P}_*^{-1}(t|t+1)\widehat{\mathbf{x}}_*(t|t+1)\,\right] \\
&= \mathbf{P}(t|N)\left[\, \mathbf{P}^{-1}(t|t-1)\widehat{\mathbf{x}}(t|t-1) + \mathbf{P}_*^{-1}(t|t)\widehat{\mathbf{x}}_*(t|t)\,\right] \\
\mathbf{P}(t|N) &= \left[\mathbf{P}^{-1}(t|t) + \mathbf{P}_*^{-1}(t|t+1)\right]^{-1} \\
&= \left[\mathbf{P}^{-1}(t|t-1) + \mathbf{P}_*^{-1}(t|t)\right]^{-1} \\
t &= 1, \ldots, N
\end{aligned}$$

where $\widehat{\mathbf{x}}_*(t|t+1)$ $[\widehat{\mathbf{x}}_*(t|t)]$ are state predictions [estimates] based on the future [present $\&$ future] measurements $\Omega_*(t+1)$ $[\Omega_*(t)]$: $\Omega_*(t) = \{\mathcal{U}_*(t), \mathcal{Y}_*(t)\}$, $\mathcal{U}_*(t) = \{\mathbf{u}(t), \ldots, \mathbf{u}(N)\}$, $\mathcal{Y}_*(t) = \{y(t), \ldots, y(N)\}$, and $\mathbf{P}_*(t|t+1)$, $\mathbf{P}_*(t|t)$ denote the corresponding covariance matrices.

All quantities needed to evaluate $\widehat{\mathbf{x}}(t|N)$ can be computed recursively using two Kalman filters/predictors: one running forwards in time, governed by the forward-time model

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{v}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t) \end{aligned}$$

and another one, designed for the backward-time system model (we assume that the state transition matrix $\mathbf{A}$ is nonsingular, i.e., invertible)

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}_*\mathbf{x}(t+1) + \mathbf{B}_*\mathbf{u}(t) + \mathbf{v}_*(t+1) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t) \end{aligned}$$

and running backwards in time.

$$\begin{aligned} \mathbf{A}_* &= \mathbf{A}^{-1} \\ \mathbf{B}_* &= -\mathbf{B}\mathbf{A}^{-1} \\ \mathbf{v}_*(t+1) &= -\mathbf{A}^{-1}\mathbf{v}(t) \\ \mathbf{V}_* &= \mathrm{cov}[\mathbf{v}_*(t)] = \mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-\mathrm{T}} \end{aligned}$$

The initial conditions for the backward Kalman filter should be set to $\widehat{\mathbf{x}}_*(N|N) = \mathbf{0}$, $\mathbf{P}_*^{-1}(N|N) = \mathbf{O}$.

Rauch-Tung-Striebel smoothing formula

Let

$$\mathbf{F}(t) = \mathbf{P}(t|t)\mathbf{A}^{\mathrm{T}}\mathbf{P}^{-1}(t+1|t).$$

The Rauch-Tung-Striebel formula can be summarized as follows

$$\begin{aligned}
\widehat{\mathbf{x}}(t|N) &= \widehat{\mathbf{x}}(t|t) + \mathbf{F}(t)\left[\widehat{\mathbf{x}}(t+1|N) - \widehat{\mathbf{x}}(t+1|t)\right] \\
\mathbf{P}(t|N) &= \mathbf{P}(t|t) + \mathbf{F}(t)[\mathbf{P}(t+1|N) - \mathbf{P}(t+1|t)]\mathbf{F}^{\mathrm{T}}(t) \\
t &= N-1, \ldots, 1
\end{aligned}$$

The initial conditions $\widehat{\mathbf{x}}_k(N|N)$ and $\mathbf{P}_k(N|N)$ are provided by the forward Kalman filter.

Bryson-Frazier smoothing formula

Let

$$\mathbf{G}(t) = \mathbf{A}[\mathbf{I} - \mathbf{K}(t)\mathbf{C}]$$

The Bryson-Frazier formula is given in the form

$$\begin{aligned}
\mathbf{r}(t-1) &= \mathbf{G}^{\mathrm{T}}(t)\mathbf{r}(t) + \mathbf{C}^{\mathrm{T}}\mathbf{S}^{-1}(t)\boldsymbol{\varepsilon}(t) \\
\mathbf{R}(t-1) &= \mathbf{G}^{\mathrm{T}}(t)\mathbf{R}(t)\mathbf{G}(t) + \mathbf{C}^{\mathrm{T}}\mathbf{S}^{-1}(t)\mathbf{C} \\
\widehat{\mathbf{x}}(t|N) &= \widehat{\mathbf{x}}(t|t-1) + \mathbf{P}(t|t-1)\mathbf{r}(t-1) \\
\mathbf{P}(t|N) &= \mathbf{P}(t|t-1) - \mathbf{P}(t|t-1)\mathbf{R}(t-1)\mathbf{P}(t|t-1) \\
t &= N-1, \ldots, 1
\end{aligned}$$

with initial conditions set to $\mathbf{r}(N) = \mathbf{0}$ and $\mathbf{R}(N) = \mathbf{O}$.

Linear quadratic Gaussian (LQG) controller
Regulator liniowo-kwadratowe-Gaussa

Assume that the controlled system admits the following state space description

$$\begin{aligned}\mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{v}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{w}(t)\end{aligned}$$

where $\{\mathbf{v}(t)\}$ and $\{\mathbf{w}(t)\}$ are zero-mean and mutually independent white Gaussian noise sequences with known covariance matrices $\mathbf{V}_{n\times n} \geq \mathbf{O}$ and $\mathbf{W}_{m\times m} > \mathbf{O}$

$$\begin{aligned}\mathbf{v}(t) &\sim \mathcal{N}(0, \mathbf{V})\\ \mathbf{w}(t) &\sim \mathcal{N}(0, \mathbf{W})\end{aligned}$$

The initial state $\mathbf{x}(0)$ is Gaussian and independent of $\{\mathbf{v}(t)\}$ and $\{\mathbf{w}(t)\}$

$$\mathbf{x}(0) \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_0), \quad \mathbf{P}_0 \geq \mathbf{O}$$

control problem

$$\mathbf{u}(0), \ldots, \mathbf{u}(N-1): \quad \mathbf{x}(0) \neq \mathbf{0} \quad \longrightarrow \quad \mathbf{x}(N) = \mathbf{0}$$

accuracy ? transient behavior ? control cost ?

Formulation of the stochastic control problem

Sformułowanie problemu sterowania stochastycznego

Find the control signal that minimizes the following cost function

$$J\left[\mathbf{u}(0),\ldots,\mathbf{u}(N-1)\right]$$
$$= \mathrm{E}\left\{\mathbf{x}^{\mathrm{T}}(N)\mathbf{Q}_0\mathbf{x}(N) + \sum_{t=0}^{N-1}\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}_1\mathbf{x}(t)\right.$$
$$\left. + \sum_{t=0}^{N-1}\mathbf{u}^{\mathrm{T}}(t)\mathbf{Q}_2\mathbf{u}(t)\right\} \longrightarrow \min$$

where $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ denote weighting matrices:

$\mathbf{Q}_0 \geq \mathbf{O}$	weighting of the terminal control error
$\mathbf{Q}_1 \geq \mathbf{O}$	weighting of the transient control error
$\mathbf{Q}_2 > \mathbf{O}$	weighting of the control effort

When we aim at stabilizing the state vector around zero (infinite control horizon), the performance measure becomes

$$\lim_{N\to\infty}\frac{1}{N}J\left[\mathbf{u}(0),\ldots,\mathbf{u}(N-1)\right]$$
$$= \mathrm{E}\left[\mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}_1\mathbf{x}(t) + \mathbf{u}^{\mathrm{T}}(t)\mathbf{Q}_2\mathbf{u}(t)\right] \longrightarrow \min$$

Control under complete state information

Sterowanie w przypadku pełnej informacji o stanie

Suppose that

$$\mathbf{y}(t) = \mathbf{x}(t)$$

Note that this is equivalent to setting $\mathbf{C} = \mathbf{I}$ and $\mathbf{W} = \mathbf{O}$.

We will look for a physically realizable control rule (*fizycznie realizowalna reguła sterowania*) $\mathbf{u}(t) = f[\mathcal{X}(t), \mathcal{U}(t-1)]$, $\mathcal{X}(t) = \{\mathbf{x}(0), \ldots, \mathbf{x}(t)\}$, $\mathcal{U}(t) = \{\mathbf{u}(0), \ldots, \mathbf{u}(t)\}$, that minimizes the quadratic performance measure.

It can be shown that the optimal control sequence $\mathbf{u}_*(t), t \in [0, N-1]$ takes the form

$$\mathbf{u}_*(t) = -\mathbf{R}(t)\mathbf{x}(t)$$

where

$$\mathbf{R}(t) = \left[\mathbf{Q}_2 + \mathbf{B}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{A}$$

and $\mathbf{H}(t)$ is the matrix evaluated recursively according to

$$\begin{aligned}\mathbf{H}(t) = \mathbf{Q}_1 + \mathbf{A}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{B} \\ \times \left[\mathbf{Q}_2 + \mathbf{B}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}(t+1)\mathbf{A}\end{aligned}$$

with the initial condition set to

$$\mathbf{H}(N) = \mathbf{Q}_0$$

Observation 1

The matrices $\mathbf{H}(N), \ldots, \mathbf{H}(0)$ must be evaluated backwards in time and memorized, i.e., it is not possible to to simultaneously update the gain matrix $\mathbf{R}(t)$ and evaluate the control signal $\mathbf{u}_*(t)$.

Observation 2

The optimal control strategy does nor depend on the value of the covariance matrix $\mathbf{V}$, i.e., it has exactly the same form as the control strategy developed for a *deterministic* system

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned}$$

under the *deterministic* measure of fit

$$\begin{aligned} & J\left[\mathbf{u}(0), \ldots, \mathbf{u}(N-1)\right] \\ & \quad = \mathbf{x}^{\mathrm{T}}(N)\mathbf{Q}_0\mathbf{x}(N) + \sum_{t=0}^{N-1} \mathbf{x}^{\mathrm{T}}(t)\mathbf{Q}_1\mathbf{x}(t) \\ & + \sum_{t=0}^{N-1} \mathbf{u}^{\mathrm{T}}(t)\mathbf{Q}_2\mathbf{u}(t) \longrightarrow \min \end{aligned}$$

Linear quadratic regulator (LQR)
Regulator liniowo-kwadratowy

When $N \to \infty$ (tracking problem) the optimal control strategy takes the form

$$\mathbf{u}_*(t) = -\mathbf{R}_\infty \mathbf{x}(t)$$

where

$$\mathbf{R}_\infty = \left[\mathbf{Q}_2 + \mathbf{B}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{A}$$

and $\mathbf{H}_\infty > \mathbf{O}$ is the positive definite solution of the following algebraic (matrix) Riccati equation

$$\begin{aligned}\mathbf{H}_\infty &= \mathbf{Q}_1 + \mathbf{A}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{B} \\ &\quad \times \left[\mathbf{Q}_2 + \mathbf{B}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{B}\right]^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{H}_\infty\mathbf{A}\end{aligned}$$

The solution of this equation exists provided that the system is controllable.

Jacopo Francesco Riccati (1676 - 1754)

Control under incomplete state information
Sterowanie w przypadku niepełnej informacji o stanie

Suppose that

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{W}, \quad \mathbf{W} > \mathbf{O}$$

We will look for a physically realizable optimal control rule of the form

$$\mathbf{u}(t) = f[\mathcal{Y}(t), \mathcal{U}(t-1)] \;\text{ or }\; \mathbf{u}(t) = f[\mathcal{Y}(t-1), \mathcal{U}(t-1)]$$

where $\mathcal{Y}(t) = \{\mathbf{y}(0), \dots, \mathbf{y}(t)\}$.

It can be shown that the optimal control sequence $\mathbf{u}_*(t), t \in [0, N-1]$ takes the form

$$\mathbf{u}_*(t) = -\mathbf{R}(t)\widehat{\mathbf{x}}(t|t) \;\text{ or }\; \mathbf{u}_*(t) = -\mathbf{R}(t)\widehat{\mathbf{x}}(t|t-1)$$

where $\mathbf{R}(t)$ denotes the control gain matrix, determined as if the full state information was available, and $\widehat{\mathbf{x}}(t|t)/\widehat{\mathbf{x}}(t|t-1)$ are state estimates yielded by the Kalman filter/predictor designed for an open-loop control system:

$$\begin{aligned}\widehat{\mathbf{x}}(t|t) &= \widehat{\mathbf{x}}(t|t-1) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}\widehat{\mathbf{x}}(t|t-1)] \\ \widehat{\mathbf{x}}(t+1|t) &= \mathbf{A}\widehat{\mathbf{x}}(t|t) + \mathbf{B}\mathbf{u}_*(t) \\ &\text{etc.}\end{aligned}$$

Separation principle
Zasada separacji

Note that

- The gain $\mathbf{R}(t)$ of the optimal controller does not depend on the covariance matrices $\mathbf{V}$ and $\mathbf{W}$ – it depends only on system matrices $(\mathbf{A}, \mathbf{B})$ and weighting matrices $(\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2)$ which characterize a deterministic control problem.

- The Kalman gain matrix $\mathbf{K}(t)$ does not depend on the control gain $\mathbf{R}(t)$ as if the state estimation was carried for an open-loop system.

The result presented above is often called a *principle of separation of estimation and control* (*zasada separacji estymacji i sterowania*) as it tells us that, under the assumptions made, the problem of designing an optimal feedback controller for a stochastic system can be broken into two separate parts, namely it can be solved by combining:

- an optimal observer designed for a stochastic system

with

- an optimal controller designed for a deterministic system

Robustness properties of LQR controllers
Odporność regulatorów LQR

When the plant is observable and controllable, the closed-loop system incorporating the LQR controller is guaranteed to be asymptotically stable.

Additionally, LQR controllers are inherently robust with respect to plant uncertainty, as they guarantee pretty satisfactory stability margins:

- positive gain margin of $+\infty$
- negative gain margin of $-\ 6$ dB
- phase margin of $\pm\ 60^\circ$

Since the LQR design procedure automatically produces controllers that are stable and robust, it is often used even if one does not really care about optimizing for energy. Moreover, this procedure is applicable to multiple-input/multiple-output plants for which classical designs are difficult to apply.

To obtain desirable properties of the closed-loop system (other than the ones mentioned above) the cost matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are usually adjusted iteratively using a trial-and-error design procedure.

Robustness properties of LQG controllers
Odporność regulatorów LQG

When the plant is observable and controllable, the closed-loop system incorporating the LQG controller is always asymptotically stable.

For minimum phase plants the same stability margins can be achieved as those guaranteed by the LQR, provided that a special design technique, known as "loop recovery", is used. Without loop recovery stability margins may be arbitrarily small.

For non-minimum phase plants no stability margins can be guaranteed.

Moving average processes
Procesy średniej ruchomej

If the noise shaping filter $H(q^{-1})$ is adopted in the form

$$H(q^{-1}) = 1 + \sum_{i=1}^{s} c_i q^{-i} = C(q^{-1})$$

i.e., it is an all-zero filter of order s, the corresponding output process is called a *moving average*

$$\mathrm{MA}(s): \quad y(t) = n(t) + \sum_{i=1}^{s} c_i n(t-i)$$

Note that $y(t)$ is a weighted sum of a finite number of past noise samples. Although the phrase “moving average” is somewhat misleading (in general the weights $1, c_1, \ldots, c_s$ do not sum to one), it is widely used in the statistical literature on time series.

Since $C(q^{-1})$ is a finite impulse response (FIR) filter, the wide sense stationarity of an MA(s) process is guaranteed for every $t > s$, irrespective of initial conditions.

Autocorrelation function of a MA process

Funkcja autokorelacji procesu MA

Let $c_0 = 1$. Straightforward calculations yield

$$0 \leq \tau \leq s :$$

$$R_y(\tau) = \mathrm{E}[y(t)y(t-\tau)] = \mathrm{E}\left\{\left[n(t) + \sum_{i=1}^{s} c_i n(t-i)\right]\right.$$

$$\left.\times \left[n(t-\tau) + \sum_{i=1}^{s} c_i n(t-\tau-i)\right]\right\}$$

$$= \sigma_n^2 \left[c_\tau + \sum_{i=1}^{s-\tau} c_i c_{\tau+i}\right]$$

$$\tau > s :$$

$$R_y(\tau) = 0$$

Unlike the AR case, the autocorrelation equations for the MA process are nonrecursive. They allow explicit computation of autocorrelation coefficients $R_y(\tau)$ given the set $c_1, \ldots, c_s$ and σ_n^2. However, solution of the inverse (identification) problem – determination of the MA coefficients based on the set of known autocorrelation coefficients – is much more difficult due to the fact that the expressions derived above are not linear in the process parameters.

Power spectrum of a MA process
Widmo mocy procesu MA

Since the MA process is a result of passing white noise through a linear all-zero filter $C(q^{-1})$, its power spectral density function can be expressed in the form

$$S_y(\omega) = \left|C(e^{-j\omega})\right|^2 \sigma_n^2 = \left|1 + \sum_{i=1}^{s} c_i e^{-j\omega i}\right|^2 \sigma_n^2$$

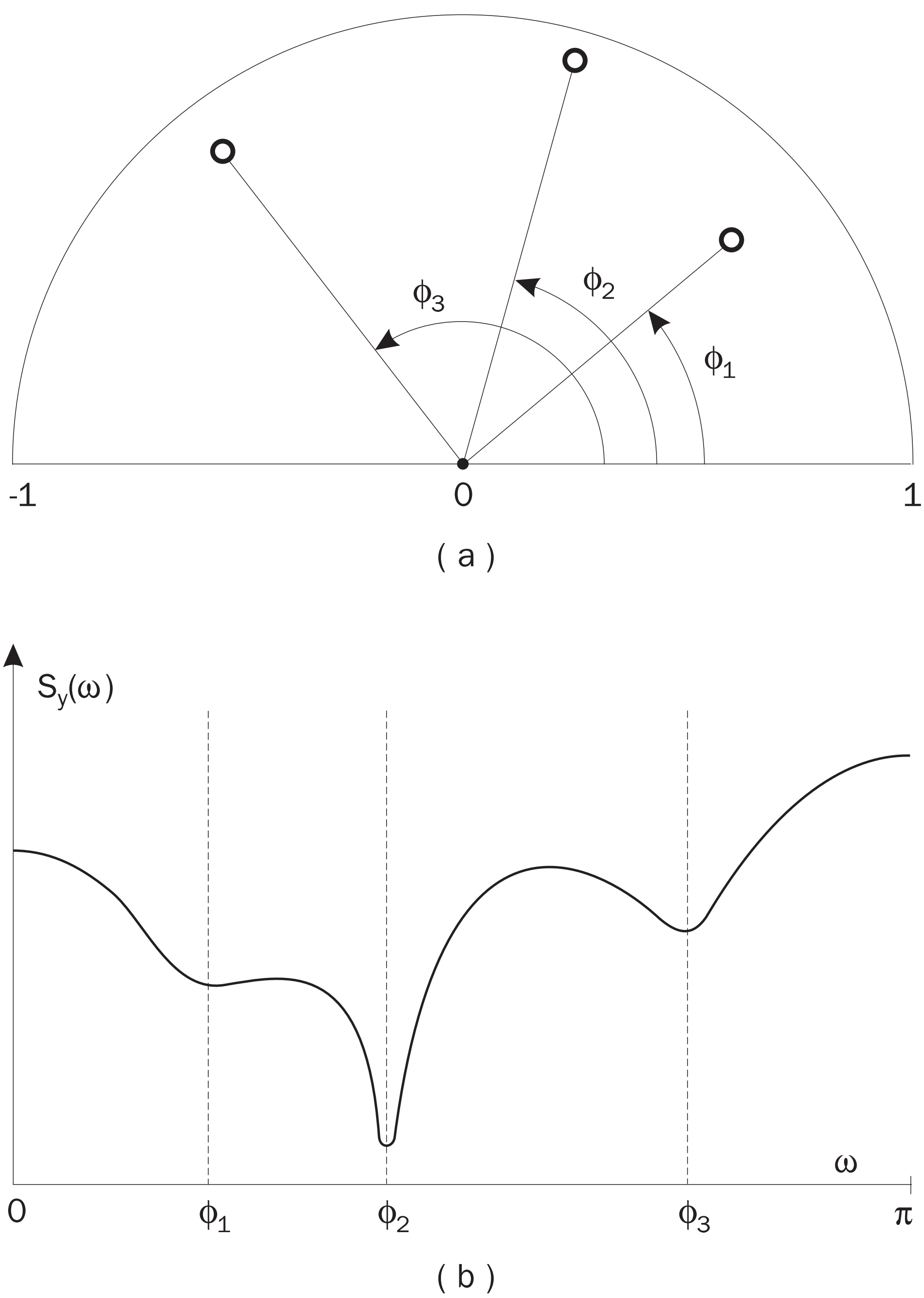
While the power spectrum of an AR process can be characterized in terms of spectral peaks (resonances), the spectrum of an MA process is in some sense composed of spectral valleys (antiresonances). Based on the distribution of zeros of $C(q^{-1})$ one can give qualitative assesment for the shape of the MA spectrum:

- Each pair of complex-conjugate zeros can introduce one spectral valley (antiresonance).

- The angular frequency coordinate of a spectral valley is approximately equal to the phase angle of the corresponding zero.

- The depth of a spectral valley is inversely proportional to the distance of the corresponding zero from the unit circle.

Like the rules for the AR case, they should be used with caution.



Approximate relationship between the location of noise shaping filter poles in the complex plane (a) and the shape of the spectral density function (b) of the corresponding MA process.

Invertibility of a MA process
Odwracalność procesu MA

Special case

Consider the MA(1) process governed by

$$y(t) = n(t) + cn(t-1)$$

Can one recover $\{n(t)\}$ based on $\{y(t)\}$?

Note that
$$n(t) = y(t) - cn(t-1)$$
Consider the following recursive "inversion" scheme

$$\widehat{n}(t) = y(t) - c\widehat{n}(t-1)$$

Let $\eta(t) = n(t) - \widehat{n}(t)$. Observe that

$$\eta(t) = -c\eta(t-1) = (-c)^t\eta(0)$$

Hence, irrespective of the initial error $\eta(0)$, it holds that

$$|c| < 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \eta(t) = 0$$

invertible MA process

Invertibility of a MA process
Odwracalność procesu MA

General case

$$n(t) = y(t) - \sum_{i=1}^{s} c_i n(t-i)$$

$$\widehat{n}(t) = y(t) - \sum_{i=1}^{s} c_i \widehat{n}(t-i)$$

$$\eta(t) = -\sum_{i=1}^{s} c_i \eta(t-i)$$

$$C(q^{-1})\eta(t) = 0$$

inveribility condition / warunek odwracalności

Denote by $q_1, \ldots, q_s$ the zeros of the forming filter $H(q^{-1})$, i.e. the roots of the polynomial $C(q^{-1})$

$$C(q_k^{-1}) = 1 + \sum_{i=1}^{s} c_i q_k^{-i} = 0, \qquad k = 1, \ldots, s$$

The MA(s) process is invertible if all roots of $C(q^{-1})$ lie inside the unit circle in the complex plane: $|q_k| < 1$, $k = 1, \ldots, s$. In such a case

$$\lim_{t \to \infty} [\widehat{n}(t) - n(t)] = 0$$

Invertibility of a MA process
Odwracalność procesu MA

What should one do if the invertibility condition is not fulfilled? The answer is somewhat surprising: every moving average process with no spectral zeros on the unit circle has an invertible representation.

<u>Theorem</u>

Consider a noninvertible MA(s) process

$$y(t) = C(q^{-1})n(t) = n(t) + \sum_{i=1}^{s} c_i n(t-i)$$

such that $C(q^{-1})$ has no zeros on the unit circle and at least one zero outside the unit circle in the complex plane. Let $\{n(t)\}$ denote a white noise sequence of variance σ_n^2.

There exists a unique invertible second-order-equivalent representation of the process defined above

$$y(t) = \widetilde{C}(q^{-1})\widetilde{n}(t) = \widetilde{n}(t) + \sum_{i=1}^{s} \widetilde{c}_i \widetilde{n}(t-i),$$

such that $\widetilde{C}(q^{-1})$ has all zeros inside the unit circle and $\{\widetilde{n}(t)\}$ is another, different from $\{n(t)\}$, white noise sequence of variance $\sigma_{\widetilde{n}}^2 > \sigma_n^2$.

Special case

Consider the MA(1) process governed by

$$y(t) = n(t) + cn(t-1)$$

Note that

$$R_y(0) = (1+c^2)\sigma_n^2, \quad R_y(1) = c\sigma_n^2$$

and hence

$$\frac{R_y(0)}{R_y(1)} = \frac{1+c^2}{c}$$

which can be rewritten as a quadratic equation with respect to c

$$c^2 - \frac{R_y(0)}{R_y(1)}\, c + 1 = 0$$

Such a quadratic equation has two solutions c and $\widetilde{c}$ which, according to the Vieta's formulas, must obey: $c \cdot \widetilde{c} = 1$

$$|c| > 1 \implies |\widetilde{c}| < 1$$

Additionally, it holds that

$$R_y(1) = \widetilde{c}\sigma_{\widetilde{n}}^2 = c\sigma_n^2 \implies \sigma_{\widetilde{n}}^2 = \frac{c}{\widetilde{c}}\,\sigma_n^2 = c^2\sigma_n^2 > \sigma_n^2$$

General case

Two zero-mean processes are second-order-equivalent if they have identical autocorrelation functions (or, equivalently, the same power spectral density functions). Therefore, to prove Theorem it is sufficient to show how to construct an invertible polynomial $\widetilde{C}(q^{-1})$ such that

$$S_y(\omega) = |\widetilde{C}(e^{-j\omega})|^2 \sigma_{\widetilde{n}}^2 = |C(e^{-j\omega})|^2 \sigma_n^2$$

Rewrite $C(q^{-1})$ in a factorized form

$$C(q^{-1}) = (1 - q_1 q^{-1}) \times \cdots \times (1 - q_s q^{-1})$$

where $q_1, \ldots, q_s$ are the zeros of $C(q^{-1})$. Note that

$$|C(e^{-j\omega})|^2 = \prod_{i=1}^{s} |1 - q_i e^{-j\omega}|^2$$

Suppose that the ith zero of $C(q^{-1})$ lies outside the unit circle, i.e. $|q_i| > 1$. Since

$$|1 - q_i e^{-j\omega}|^2 = |q_i|^2 \left| 1 - \frac{1}{q_i^\star} \, e^{-j\omega} \right|^2$$

one can rewrite $|C(e^{-j\omega})|^2$ in the form

$$|C(e^{-j\omega})|^2 = \delta |\widetilde{C}(e^{-j\omega})|^2$$

where

$$\widetilde{C}(q^{-1}) = \prod_{i=1}^{s}(1 - \widetilde{q}_i q^{-1})$$

$$\widetilde{q}_i = \begin{cases} 1/q_i^\star & \text{if} \quad |q_i| > 1 \\ q_i & \text{if} \quad |q_i| < 1 \end{cases}$$

and

$$\delta = \prod_{i=1}^{s} \delta_i, \quad \delta_i = \begin{cases} |q_i|^2 & \text{if} \quad |q_i| > 1 \\ 1 & \text{if} \quad |q_i| < 1 \end{cases}$$

Observe that all zeros of $\widetilde{C}(q^{-1})$ lie inside the unit circle and that $\delta > 1$, since at least one zero of $C(q^{-1})$ was assumed to lie outside the unit circle. Finally, note that putting

$$\sigma_{\widetilde{n}}^2 = \delta \sigma_n^2 > \sigma_n^2$$

one can rewrite $C(q^{-1})n(t)$ as $\widetilde{C}(q^{-1})\widetilde{n}(t)$, which completes our construction of an invertible MA representation.

All representations are equivalent in the sense that they characterize stochastic processes with an identical covariance structure. Even though invertible and noninvertible models can be used to generate an MA process, only the invertible model can be effectively used to predict its future. Consequently, the invertible MA representation is the only one that has a practical significance. Note that the driving noise variance, and hence also the limiting value of the mean square one-step-ahead prediction error, takes the largest value for the invertible process representation.

Equivalence of AR and MA models
Równoważność modeli AR i MA

Theorem

Every finite-order wide-sense stationary autoregressive process can be described by an invertible infinite-order moving average model and vice versa.

$$T(q^{-1})y(t) = n(t) \quad \Longleftrightarrow \quad y(t) = \frac{1}{T(q^{-1})}\, n(t)$$

Example 1

$$\mathrm{AR}(1): \quad y(t) = ay(t-1) + n(t), \quad |a| < 1$$

$$\Downarrow$$

$$(1 - aq^{-1})y(t) = n(t)$$

$$\Downarrow$$

$$y(t) = \frac{1}{1 - aq^{-1}}\, n(t)$$

$$\frac{1}{1 - aq^{-1}} = 1 + \sum_{i=1}^{\infty} a^i q^{-i}$$

$$\Downarrow$$

$$\mathrm{MA}(\infty): \quad y(t) = n(t) + \sum_{i=1}^{\infty} a^i n(t-i)$$

Example 2

$$\mathrm{MA}(1): \quad y(t) = n(t) + cn(t-1), \quad |c| < 1$$

$$\Downarrow$$

$$y(t) = (1 + cq^{-1})n(t)$$

$$\Downarrow$$

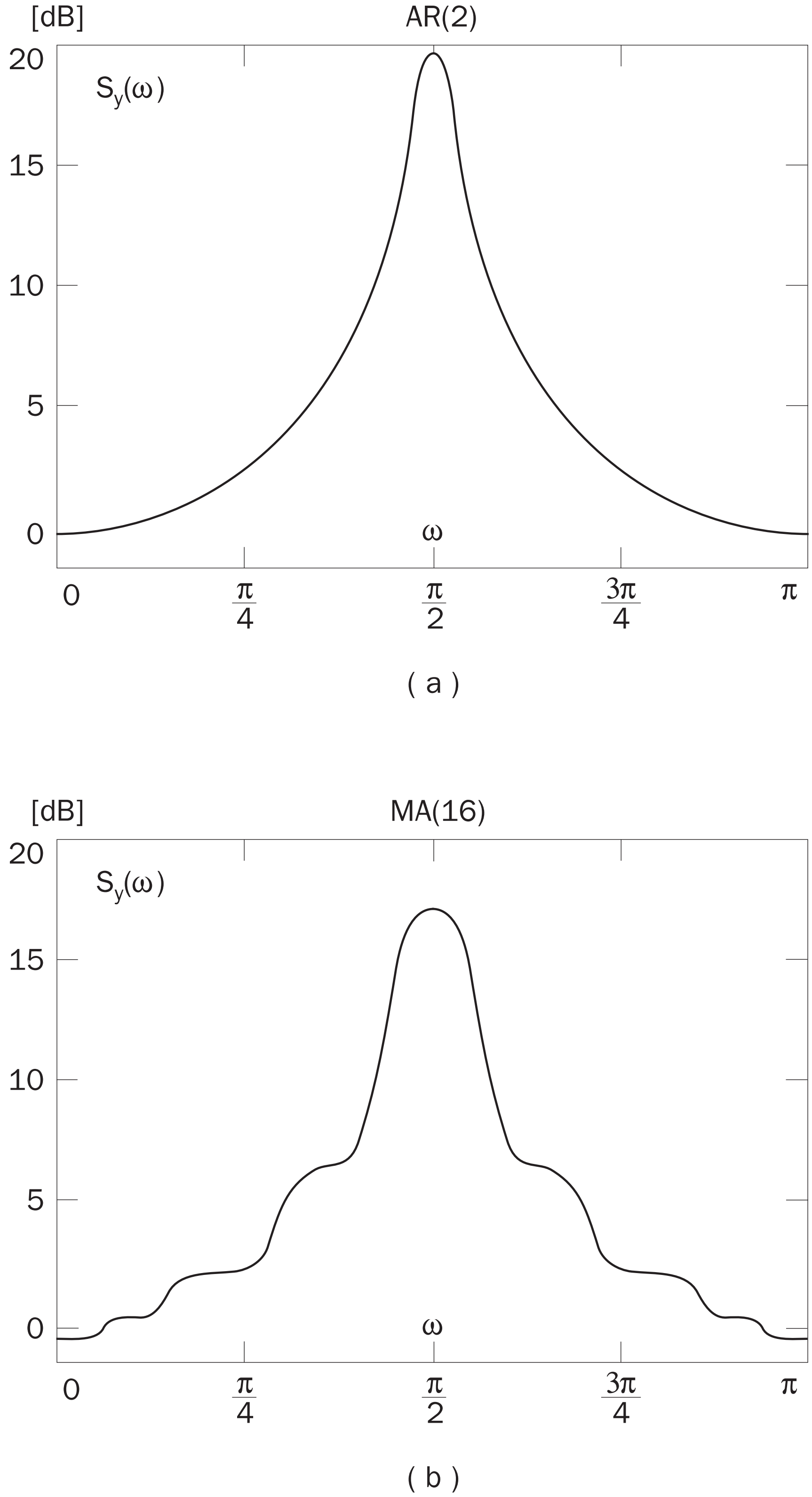
$$\frac{1}{1+cq^{-1}}\, y(t) = n(t)$$

$$\frac{1}{1+cq^{-1}} = 1 + \sum_{i=1}^{\infty}(-c)^i q^{-i}$$

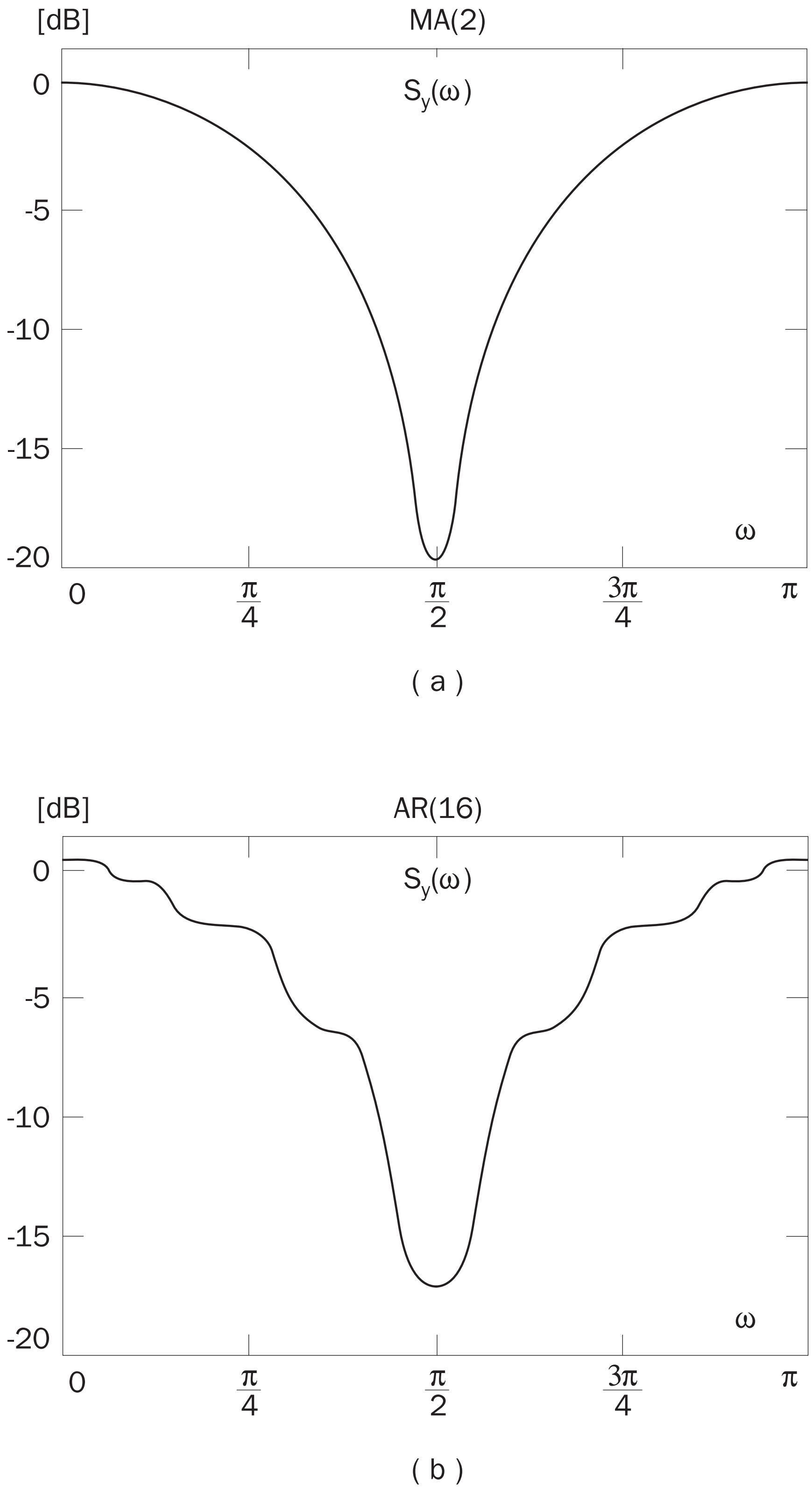
$$\Downarrow$$

$$\mathrm{AR}(\infty): \quad y(t) = -\sum_{i=1}^{\infty}(-c)^i y(t-i) + n(t)$$

In view of the asymptotic equivalence of AR and MA models, should one care which modeling option to choose? According to the principle of parsimony certainly yes. Processes that can be satisfactorily approximated by low-order autoregressive models may require tens of MA parameters to reach a comparable degree of approximation and vice versa. A good trade-off between the approximation quality and the model complexity can usually be achieved by combining in one description both autoregressive and moving average terms; this is the so-called ARMA model.



Power spectrum of a second-order AR process (a) and its 16-th order MA approximation.



Power spectrum of a second-order MA process (a) and its 16-th order AR approximation.

Mixed autoregressive moving average process

Proces mieszany autoregresji - średniej ruchomej

The mixed autoregressive moving average (ARMA) model combines both AR and MA terms:

$$y(t) = \sum_{i=1}^{r} a_i y(t-i) + n(t) + \sum_{i=1}^{s} c_i n(t-i)$$

which is equivalent to adopting the following noise shaping filter with both poles and zeros:

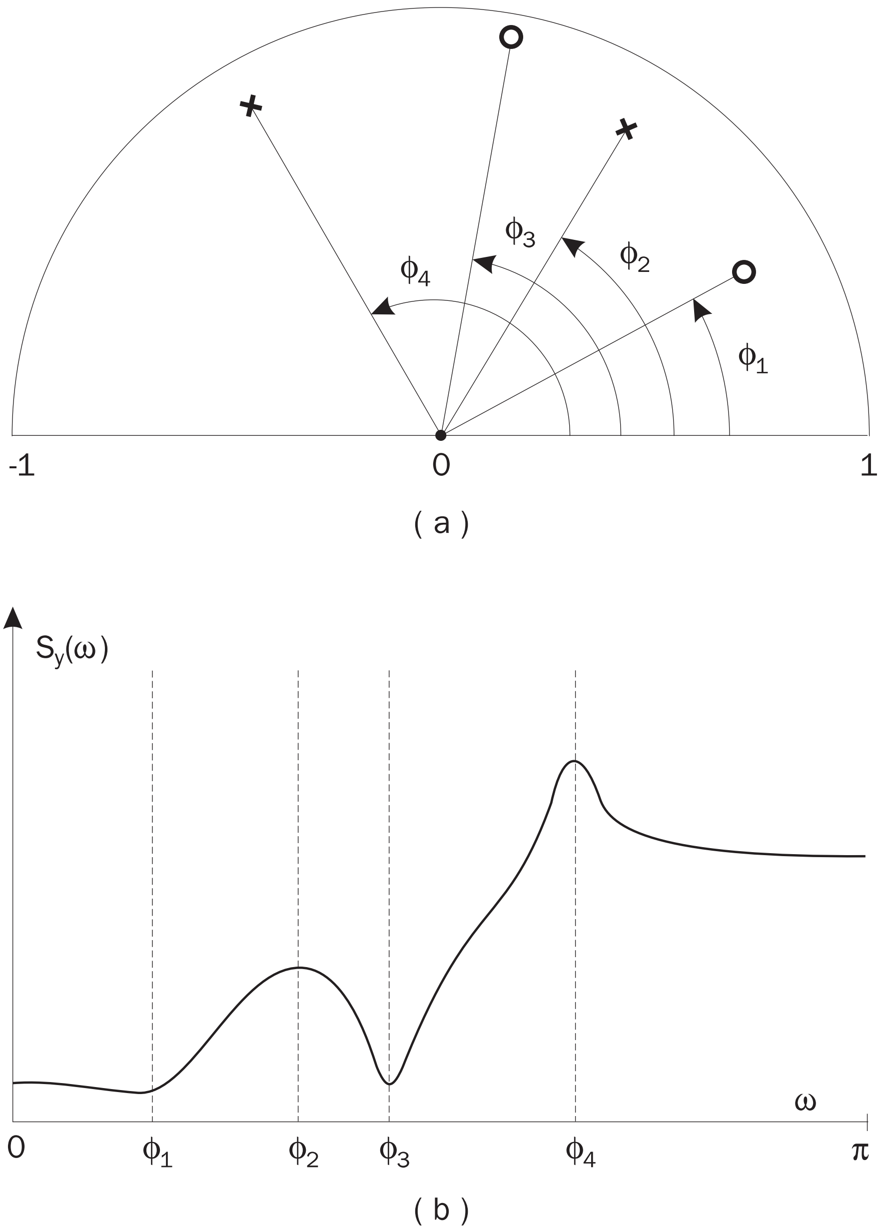
$$H(q^{-1}) = \frac{C(q^{-1})}{A(q^{-1}} = \frac{1 + \sum_{i=1}^{s} c_i q^{-i}}{1 - \sum_{i=1}^{r} a_i q^{-i}}$$

The ARMA process is asymptotically stationary if all zeros of $A(q^{-1})$ lie inside the unit circle in the complex plane; it is invertible if its moving average part is invertible, i.e. if all zeros of $C(q^{-1})$ lie inside the unit circle.

The spectral density function of an ARMA signal can be expressed in the form

$$S_y(\omega) = \frac{|C(e^{-j\omega})|^2}{|A(e^{-j\omega})|^2}\,\sigma_n^2 = \frac{|1 + \sum_{i=1}^{s} c_i e^{-j\omega i}|^2}{|1 - \sum_{i=1}^{r} a_i e^{-j\omega i}|^2}\,\sigma_n^2$$

Since the noise shaping filter $H(q^{-1})$ has both poles and zeros, the spectrum of an ARMA process can easily match the spectral peaks and spectral valleys of the modeled signal



Approximate relationship between the location of shaping filter poles and zeros in the complex plane (a) and the shape of the spectral density function (b) of the corresponding ARMA process.

ARX and ARMAX models of controlled systems

Modele ARX i ARMAX obiektów sterowania

ARX – autoregressive with exogenous input

$$y(t) = \sum_{i=1}^{r} a_i y(t-i) + \sum_{i=0}^{p} b_i u(t-i-k) + n(t)$$

$$A(q^{-1})y(t) = B(q^{-1})u(t-k) + n(t)$$

where

$$B(q^{-1}) = \sum_{i=0}^{p} b_i q^{-i}$$

and $k \geq 1$ denotes *transport delay* (*opóźnienie transporto-we*).

ARMAX – autoregressive moving average with exogenous input

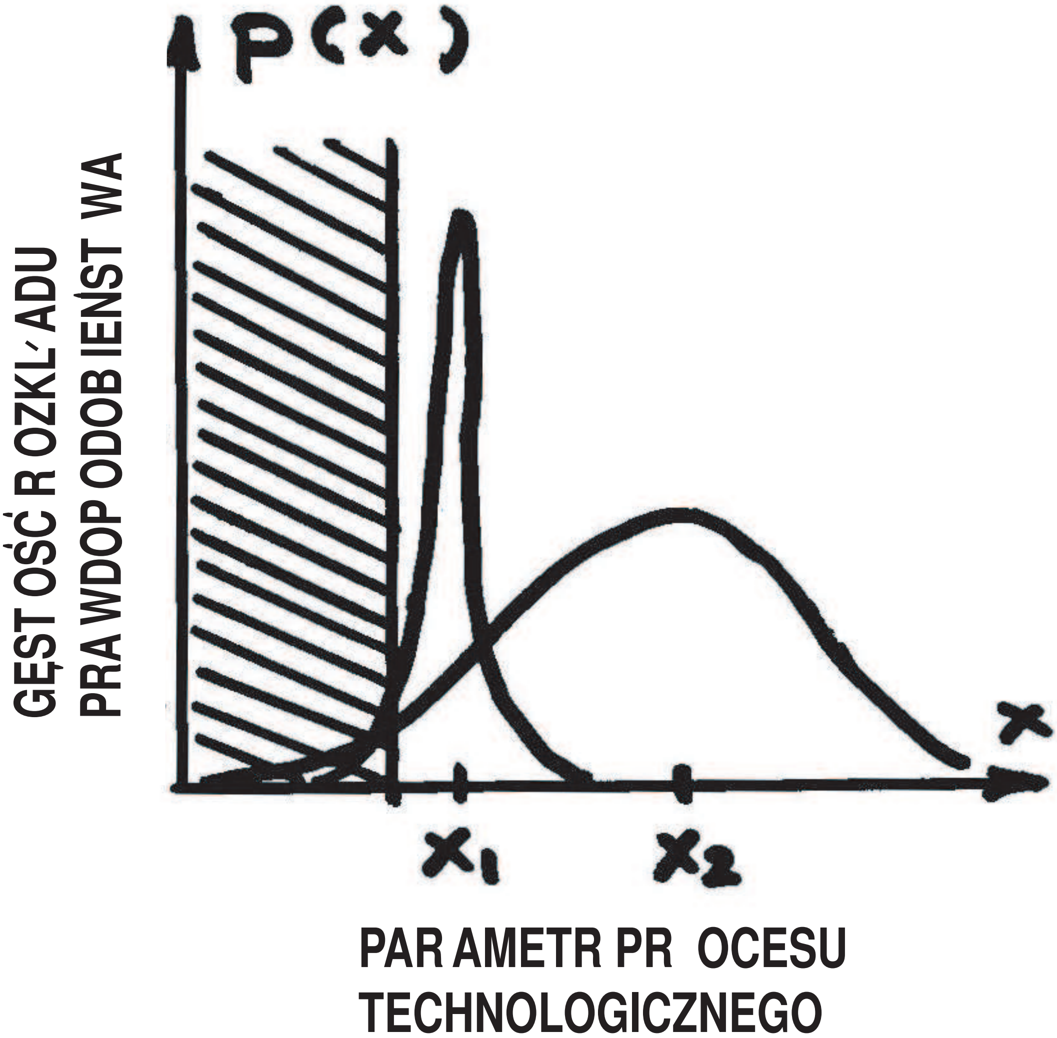
$$y(t) = \sum_{i=1}^{r} a_i y(t-i) + \sum_{i=0}^{p} b_i u(t-i-k)$$

$$+ n(t) + \sum_{i=1}^{s} c_i n(t-i)$$

$$A(q^{-1})y(t) = B(q^{-1})u(t-k) + C(q^{-1})n(t)$$

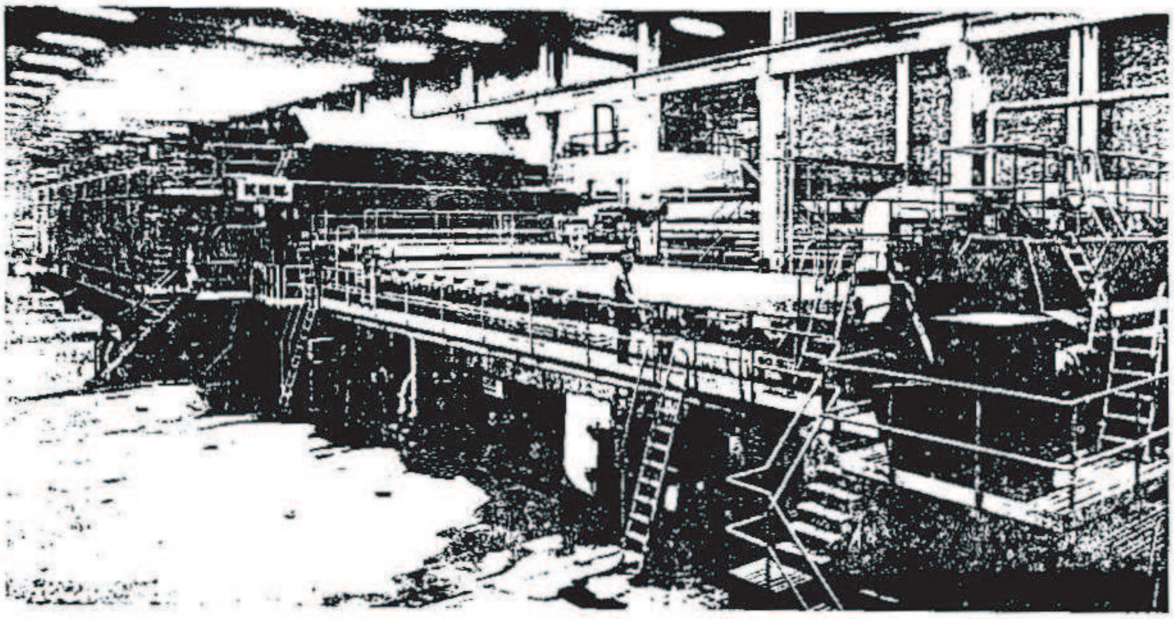
Minimum-variance control

Sterowanie minimalnowariancyjne

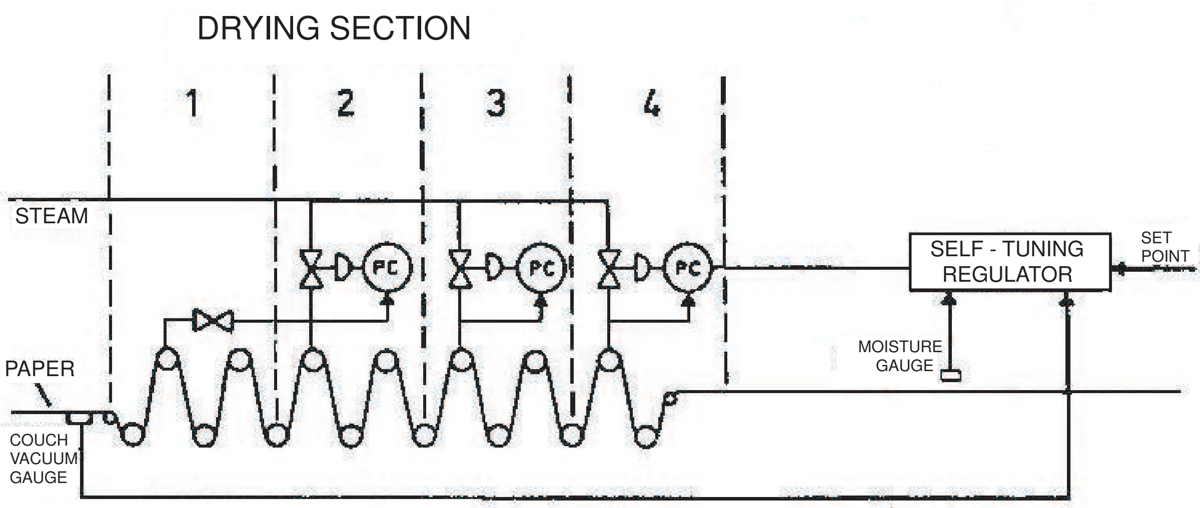


$$u(t): \quad \mathrm{var}[x(t)] \quad \longrightarrow \quad \min$$

Example



Paper production plant owned by
the Billerund company (Sweden)
130,000 t/year



The steam drying section of the paper production line.

Minimum-variance control
Sterowanie minimalnowariancyjne

Introductory example 1

Consider the problem of minimum-variance control of an ARX plant with unity delay ($k = 1$)

$$y(t+1) = \sum_{i=1}^{r} a_i y(t+1-i) + b_0 u(t) + \sum_{i=1}^{p} b_i u(t-i) + n(t+1)$$

$$u(t): \quad \mathrm{E}[y^2(t)] \longrightarrow \min$$

$$\Downarrow$$

$$u(t) = -\frac{1}{b_0}\left[\sum_{i=1}^{r} a_i y(t+1-i) + \sum_{i=1}^{p} b_i u(t-i)\right] = u_{\mathrm{MV}}(t)$$

$$\Downarrow$$

$$y(t) = n(t), \quad \sigma_y^2 = \sigma_n^2$$

Invertibility of ARMAX models

Odwracalność modeli ARMAX

Consider an ARMAX model

$$A(q^{-1})y(t) = B(q^{-1})u(t-k) + C(q^{-1})n(t)$$

and let

$$\widehat{n}(t) = -\sum_{i=1}^{s} c_i \widehat{n}(t-i) + y(t) - \sum_{i=1}^{r} a_i y(t-i) - \sum_{i=0}^{p} b_i u(t-i-k)$$

with arbitrary initial conditions $\widehat{n}(1), \ldots, \widehat{n}(s)$.

Note that this recursive scheme can be compactly written down in the form

$$C(q^{-1})\widehat{n}(t) = A(q^{-1})y(t) - B(q^{-1})u(t-k)$$

<u>Fact 1</u>

When the ARMAX model is invertible, i.e., when if all zeros of $C(q^{-1})$ lie inside the unit circle in the complex plane, it holds that

$$\lim_{t\to\infty} [\widehat{n}(t) - n(t)] = 0$$

<u>Fact 2</u>

Every noninverible ARMAX model has its invertible representation.

Minimum-variance control
Sterowanie minimalnowariancyjne

Introductory example 2

Consider the problem of minimum-variance control of the following ARMAX plant

$$y(t) = ay(t-1) + bu(t-1) + n(t) + cn(t-1)$$

$$\begin{aligned} A(q^{-1}) &= 1 - aq^{-1} \\ B(q^{-1}) &= b \\ C(q^{-1}) &= 1 + cq^{-1} \end{aligned}$$

$$y(t+1) = \frac{b}{1-aq^{-1}}\, u(t) + \frac{1+cq^{-1}}{1-aq^{-1}}\, n(t+1)$$

Since

$$\frac{1+cq^{-1}}{1-aq^{-1}} = 1 + \frac{(c+a)q^{-1}}{1-aq^{-1}}$$

one obtains

$$y(t+1) = \frac{b}{1-aq^{-1}}\, u(t) + n(t+1) + \frac{(c+a)}{1-aq^{-1}}\, n(t)$$

Minimum-variance control
Sterowanie minimalnowariancyjne

Requiring that

$$\frac{b}{1-aq^{-1}}\,u(t)+\frac{(c+a)}{1-aq^{-1}}\,n(t)=0$$

one arrives at

$$u(t)=-\,\frac{c+a}{b}\,n(t),\quad y(t)=n(t)$$

which finally leads to

$$u(t)=-\,\frac{c+a}{b}\,y(t)=u_{\mathrm{MV}}(t)$$

Note that the MV controller for this plant can be alternatively put down in the form

$$\begin{aligned}u(t)&=-\,\frac{1}{b}\,\left[ay(t)+c\widehat{n}(t)\right]\\ \widehat{n}(t)&=-c\widehat{n}(t-1)+y(t)\\ &\qquad -ay(t-1)-bu(t-1)\end{aligned}$$

Minimum-variance control
Sterowanie minimalnowariancyjne

General case

Consider the problem of minimum-variance control of the following *invertible* ARMAX plant

$$A(q^{-1})y(t) = B(q^{-1})u(t-k) + C(q^{-1})n(t)$$

where (for simplicity) $\deg(A) = \deg(B) = \deg(C) = r$:

$$A(q^{-1}) = 1 - \sum_{i=1}^{r} a_i q^{-i}$$

$$B(q^{-1}) = \sum_{i=0}^{r} b_i q^{-i}$$

$$C(q^{-1}) = 1 + \sum_{i=1}^{r} c_i q^{-i}$$

Note that

$$y(t+k) = \frac{B(q^{-1})}{A(q^{-1})}\, u(t) + \frac{C(q^{-1})}{A(q^{-1})}\, n(t+k)$$

Denote by $F(q^{-1})$ and $G(q^{-1})$ the polynomials that obey the following Diophantine equation (*równanie diofantyczne*)

$$\frac{C(q^{-1})}{A(q^{-1})} = F(q^{-1}) + q^{-k}\,\frac{G(q^{-1})}{A(q^{-1})}$$

where $\deg(F) = k-1$, $\deg(G) = r-1$:

$$F(q^{-1}) = 1 + \sum_{i=1}^{k-1} f_i q^{-i}$$

$$G(q^{-1}) = \sum_{i=0}^{r-1} g_i q^{-i}$$

$$y(t+k) = F(q^{-1})n(t+k) + \frac{B(q^{-1})}{A(q^{-1})}\,u(t) + \frac{G(q^{-1})}{A(q^{-1})}\,n(t)$$

$$n(t) = \frac{A(q^{-1})}{C(q^{-1})}\,y(t) - q^{-k}\,\frac{B(q^{-1})}{C(q^{-1})}\,u(t)$$

$$y(t+k) = F(q^{-1})n(t+k) + \frac{G(q^{-1})}{C(q^{-1})}\,y(t) + \left[\frac{B(q^{-1})}{A(q^{-1})} - q^{-k}\,\frac{B(q^{-1})G(q^{-1})}{A(q^{-1})C(q^{-1})}\right]u(t)$$

$$\frac{C}{A} = F + q^{-k}\,\frac{G}{A}$$

$$\Downarrow$$

$$\frac{B}{A} - q^{-k}\,\frac{BG}{AC} = \frac{B}{C}\,\left[\frac{C}{A} - q^{-k}\,\frac{G}{A}\right] = \frac{BF}{C}$$

$$y(t+k) = F(q^{-1})n(t+k) + \frac{G(q^{-1})}{C(q^{-1})}\,y(t) + \frac{B(q^{-1})F(q^{-1})}{C(q^{-1})}\,u(t)$$

$$\mathrm{var}[y(t+k)] = \mathrm{E}[y^2(t+k)] = E\left\{\left[F(q^{-1})n(t+k)\right]^2\right\} + E\left\{\left[\frac{G(q^{-1})}{C(q^{-1})}\,y(t) + \frac{B(q^{-1})F(q^{-1})}{C(q^{-1})}\,u(t)\right]^2\right\} \to \min$$

$$\Downarrow$$

$$u(t) = -\,\frac{G(q^{-1})}{B(q^{-1})F(q^{-1})}\,y(t) = u_{\mathrm{MV}}(t)$$

Stability and performance of a MV controller
Stabilność i jakość sterowania regulatora MV

When $u(t) = u_{\mathrm{MV}}(t)$ it holds that

$$y(t) = F(q^{-1})n(t)$$

$$u(t) = -\frac{G(q^{-1})}{B(q^{-1})}\,n(t)$$

Stability condition

The MV regulator is stable if all roots of $B(q^{-1})$ lie inside the unit circle in the complex plane, i.e., if the controlled system is minimum phase (*obiekt minimalnofazowy*).

$$\begin{aligned} y(t) = F(q^{-1})n(t) &= n(t) \\ &+ f_1 n(t-1) + \ldots + f_{k-1}n(t-k+1) \end{aligned}$$

$$\mathrm{var}[y(t)] = \mathrm{E}[y^2(t)] = (1 + f_1^2 + \ldots + f_{k-1}^2)\sigma_n^2$$

Example 1

Suppose that

$$A(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2}$$
$$B(q^{-1}) = 1 + 0.5q^{-1}$$
$$C(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}$$

$$k = 1$$

$$C(q^{-1}) = A(q^{-1})F(q^{-1}) + q^{-1}G(q^{-1})$$
$$F(q^{-1}) = 1$$

$$\begin{aligned} 1 + 1.5q^{-1} + 0.9q^{-2} &= 1 - 1.7q^{-1} + 0.7q^{-2} \\ &+ q^{-1}(g_0 + g_1 q^{-1}) \end{aligned}$$

$$\left\{ \begin{array}{rcr} 1.5 & = & -1.7 + g_0 \\ 0.9 & = & 0.7 + g_1 \end{array} \right. \quad \Rightarrow \quad g_0 = 3.2, \;\; g_1 = 0.2$$

$$u(t) = -\frac{G(q^{-1})}{B(q^{-1})F(q^{-1})} y(t) = -\frac{3.2 + 0.2q^{-1}}{1 + 0.5q^{-1}} y(t)$$

$$u(t) = -0.5u(t-1) - 3.2y(t) - 0.2y(t-1)$$

$$y(t) = n(t)$$

Example 2

Suppose that

$$A(q^{-1}) = 1 - 1.7q^{-1} + 0.7q^{-2}$$
$$B(q^{-1}) = 1 + 0.5q^{-1}$$
$$C(q^{-1}) = 1 + 1.5q^{-1} + 0.9q^{-2}$$

$$k = 2$$

$$C(q^{-1}) = A(q^{-1})F(q^{-1}) + q^{-1}G(q^{-1})$$
$$F(q^{-1}) = 1 + f_1 q^{-1}$$

$$\begin{aligned} 1 + 1.5q^{-1} + 0.9q^{-2} \\ &= (1 - 1.7q^{-1} + 0.7q^{-2})(1 + f_1 q^{-1}) \\ &+ q^{-2}(g_0 + g_1 q^{-1}) \end{aligned}$$

$$\left\{ \begin{array}{rcl} 1.5 & = & -1.7 + f_1 \\ 0.9 & = & 0.7 - 1.7f_1 + g_0 \\ 0 & = & 0.7f_1 + g_1 \end{array} \right. \quad \Rightarrow \quad \begin{array}{l} f_1 = 3.2 \\ g_0 = 5.64 \\ g_1 = -2.24 \end{array}$$

$$\begin{aligned} u(t) &= -3.7u(t-1) - 1.6u(t-2) \\ &- 5.64y(t) + 2.24y(t-1) \end{aligned}$$

$$y(t) = n(t) + 3.2n(t-1)$$

Linear Diophantine equations
Liniowe równania diofantyczne

Diophantus of Alexandria (III A.C) ...

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO V.C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobaea, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

... and his book

Given $a, b, c \in C$ find all solutions of the equation

$$ax + by = c$$

such that $x, y \in C$

Linear Diophantine equations
Liniowe równania diofantyczne

Example

$$3x + 5y = 8$$

This equation has infinitely many solutions of the form

$$x = x_0 + 5d, \quad y = y_0 - 3d$$

where $d \in C$ denotes *any* integer number and

$$x_0 = 1, \quad y_0 = 1$$

denotes the so-called minimal solution.

Solution

Diophantine equation $ax + by = c$ has solution iff (if and only if) c is a multiple of the greatest common divisor of a and b. The solution has the form

$$x = x_0 + bd, \quad y = y_0 - ad$$

where $\{x_0, y_0\}$ is the minimal solution and $d \in C$ is any integer number.

Polynomial Diophantine equations
Wielomianowe równania diofantyczne

The polynomial Diophantine equation

$$A(q^{-1})X(q^{-1}) + B(q^{-1})Y(q^{-1}) = C(q^{-1})$$

has solution iff $C(q^{-1})$ is a multiple of the greatest common divisor of $A(q^{-1})$ and $B(q^{-1})$. The solution has the form

$$X(q^{-1}) = X_0(q^{-1}) + B(q^{-1})D(q^{-1})$$
$$Y(q^{-1}) = Y_0(q^{-1}) - A(q^{-1})D(q^{-1})$$

where

$$X_0(q^{-1}), \quad Y_0(q^{-1})$$

is the minimal solution and $D(q^{-1})$ is any polynomial.

Excercise

Find all solutions (if any) of the polynomial Diophantine equation in the case where

$$A(q^{-1}) = 1 + 0.7q^{-1} + 0.1q^{-1}$$
$$B(q^{-1}) = 1 + 0.9q^{-1} + 0.2q^{-1}$$
$$C(q^{-1}) = 1 + 0.3q^{-1} - 0.1q^{-1}$$

Minimum-variance tracking
Śledzenie minimalnowariancyjne

$$A(q^{-1})y(t+k) = B(q^{-1})u(t) + C(q^{-1})n(t+k)$$

$$u(t): \quad \mathrm{E}\left\{[y(t) - y_r(t)]^2\right\} \longrightarrow \min$$

$y_r(t)$ – reference signal (*sygnał zadający*)

Suppose that the controlled system is minimum phase, and that at each time instant t we know k future values of the reference signal: $y_r(t+1), \ldots, y_r(t+k)$.

Denote by $u_r(t)$ the signal that obeys

$$A(q^{-1})y_r(t+k) = B(q^{-1})u_r(t)$$

Observe that

$$A(q^{-1})\left[y(t+k) - y_r(t+k)\right] = B(q^{-1})\left[u(t) - u_r(t)\right] + C(q^{-1})n(t+k)$$

Hence, the minimum-variance tracker takes the form

$$u(t) - u_r(t) = -\frac{G(q^{-1})}{B(q^{-1})F(q^{-1})}\left[y(t) - y_r(t)\right]$$

$$C(q^{-1}) = A(q^{-1})F(q^{-1}) + q^{-k}G(q^{-1})$$

Minimum-variance tracking
Śledzenie minimalnowariancyjne

$$u(t) = -\frac{G(q^{-1})}{B(q^{-1})F(q^{-1})}\,y(t) + \frac{1}{B(q^{-1})}\left[A(q^{-1}) + q^{-k}\,\frac{G(q^{-1})}{F(q^{-1})}\right] y_r(t+k)$$

$$A(q^{-1}) + q^{-k}\,\frac{G(q^{-1})}{F(q^{-1})} = \frac{C(q^{-1})}{F(q^{-1})}$$

$$u(t) = -\frac{G(q^{-1})}{B(q^{-1})F(q^{-1})}\,y(t) + \frac{C(q^{-1})}{B(q^{-1})F(q^{-1})}\,y_r(t+k)$$

tracking accuracy / dokładność śledzenia

$$y(t) - y_r(t) = F(q^{-1})n(t)$$

Drawbacks of MV controllers
Wady regulatorów MV

1. Minimum-variance control is the "cheap control" strategy, which does not account for the control effort. In many applications a more adequate control objective is to minimize $\mathrm{E}[y^2(t)] + \mu\mathrm{E}[u^2(t)]$.

2. The MV controllers can be applied *only* to minimum phase plants.

   For sufficiently large sampling frequencies, i.e., for sufficiently small values of T_s, the discrete-time model

   $$G_d(z) = \mathcal{Z}\left\{\mathcal{L}^{-1}\left[\frac{1-e^{-sT_s}}{s}\,G(s)\right]\right\}$$

   obtained by means of discretization of a linear continuous-time rational system with transfer function

   $$G(s) = \frac{B(s)}{A(s)}\,, \quad \deg(A) \geq \deg(B) + 3$$

   is *always* non-minimum phase.

3. In some (rare) cases the MV controllers may produce intersample ripple of the *continuous-time* signal observed at the output of the controlled plant.

About zeros that should not be cancelled
O zerach, których nie należy "skracać"

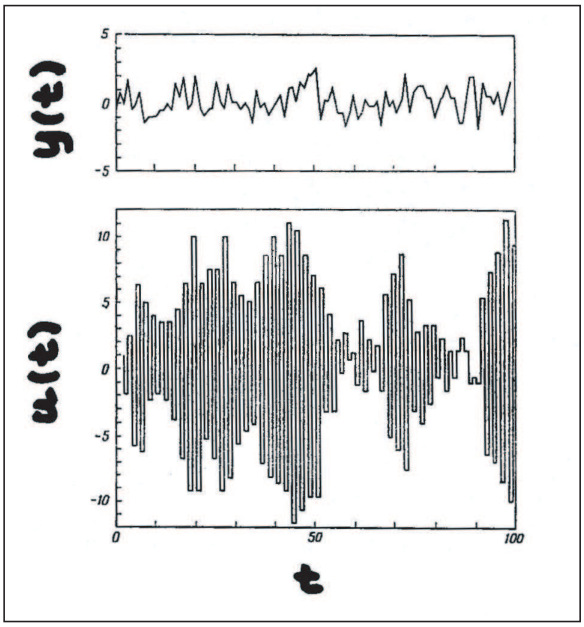
<u>Example</u>

Consider the following MV design

$$
\begin{aligned}
y(t) &= ay(t-1) + u(t-1) + bu(t-2) \\
&\quad + n(t) + cn(t-1) \\
&a = 0.7, \quad b = 0.99, \quad c = 0.95
\end{aligned}
$$

$$1 + cq^{-1} = 1 - aq^{-1} + gq^{-1} \quad \Longrightarrow \quad g = a + c$$

$$u(t) = -\frac{g}{1 + bq^{-1}}\,y(t) = -\frac{1.65}{1 + 0.99q^{-1}}\,y(t)$$



ringing (*dzwonienie*) – oscillations with period twice the sampling period

MA controller

Regulator MA

$$B(q^{-1}) = B^-(q^{-1})B^+(q^{-1})$$

$$\deg(B) = r, \quad \deg(B^-) = m, \quad \deg(B^+) = l$$

$$r = m + l$$

where

$$B^-(q^{-1}) = 1 + \beta_1 q^{-1} + \ldots + \beta_m q^{-m}$$

m zeros that should not be cancelled:
unstable + stable but too close to the
point $(-1 + j0)$ in the complex plane

$$B^+(q^{-1}) = \gamma_0 + \gamma_1 q^{-1} + \ldots + \gamma_l q^{-l}$$

l zeros that will be cancelled

MA controller
Regulator MA

Denote by $F^+(q^{-1})$ and $G^+(q^{-1})$ the polynomials that obey the following Diophantine equation

$$\frac{C(q^{-1})}{A(q^{-1})} = F^+(q^{-1}) + q^{-k}\,\frac{B^-(q^{-1})G^+(q^{-1})}{A(q^{-1})}$$

where $\deg(F^+) = k + m - 1$, $\deg(G^+) = r - 1$:

$$F^+(q^{-1}) = 1 + \sum_{i=1}^{k-1} f_i q^{-i} + \sum_{i=k}^{k+m-1} f_i^+ q^{-i}$$

$$G^+(q^{-1}) = \sum_{i=0}^{r-1} g_i^+ q^{-i}$$

Note that

$$\begin{aligned} y(t+k) &= \frac{B(q^{-1})}{A(q^{-1})}\,u(t) + \frac{G(q^{-1})}{A(q^{-1})}\,n(t) \\ &\quad + F(q^{-1})n(t+k) \\ &= \frac{B(q^{-1})}{A(q^{-1})}\,u(t) + \frac{B^-(q^{-1})G^+(q^{-1})}{A(q^{-1})}\,n(t) \\ &\quad + F^+(q^{-1})n(t+k) \end{aligned}$$

MA controller
Regulator MA

$$u_{\mathrm{MV}}(t) = -\,\frac{G(q^{-1})}{B(q^{-1})}\,n(t)$$

$$u_{\mathrm{MA}}(t) = -\,\frac{G^{+}(q^{-1})}{B^{+}(q^{-1})}\,n(t)$$

$$u(t) = -\,\frac{G^{+}(q^{-1})}{B^{+}(q^{-1})F^{+}(q^{-1})}\,y(t) = u_{\mathrm{MA}}(t)$$

It can be shown that the MV controller for a non-minimum phase plant is identical with the MA controller. The name stems from the fact that when the MA controller is applied, one obtains

$$y(t) = F^{+}(q^{-1})n(t) = n(t) + \sum_{i=1}^{k-1} f_i n(t-i) + \sum_{i=k}^{k+m-1} f_i^{+} n(t-i) = \mathrm{MA}(k+m-1)$$

i.e., the output of the controlled plant is a moving average process even if there is no transport delay ($k = 1$).

The price for not cancelling some of the system zeros is paid in performance degradation.

MA controller
Regulator MA

Example (continued)

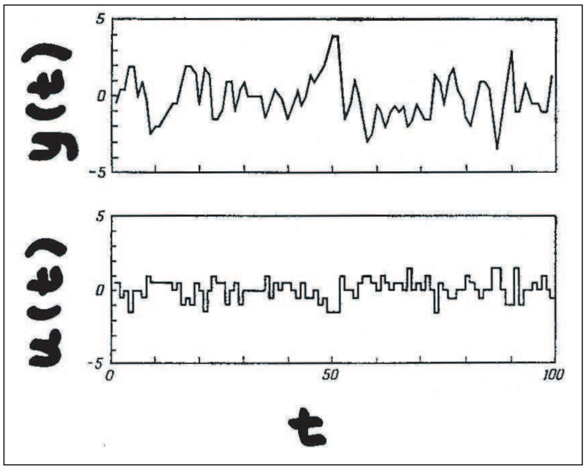
Consider the following MA design

$$
\begin{aligned}
y(t) &= ay(t-1) + u(t-1) + bu(t-2) \\
&\quad + n(t) + cn(t-1) \\
&\quad a = 0.7, \quad b = 0.99, \quad c = 0.95
\end{aligned}
$$

$$B^{-}(q^{-1}) = 1 + bq^{-1}$$

$$1 + cq^{-1} = (1 - aq^{-1})(1 + f^{+}q^{-1}) + q^{-1}(1 + bq^{-1})g^{+}$$

$$u(t) = -\frac{g^{+}}{1 + f^{+}q^{-1}}\, y(t) = -\frac{0.46}{1 + 0.65q^{-1}}\, y(t)$$



THE END