A 27/26-Approximation Algorithm for the Chromatic Sum Coloring of Bipartite Graphs

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Abstract. We consider the Chromatic Sum Problem on bipartite graphs which appears to be much harder than the classical Chromatic Number Problem. We prove that the Chromatic Sum Problem is NP-complete on planar bipartite graphs with $\Delta \leq 5$, but polynomial on bipartite graphs with $\Delta \leq 3$, for which we construct an $O(n^2)$-time algorithm. Hence, we tighten the borderline of intractability for this problem on bipartite graphs with bounded degree, namely: the case $\Delta = 3$ is easy, $\Delta = 5$ is hard. Moreover, we construct a 27/26-approximation algorithm for this problem thus improving the best known approximation ratio of 10/9.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. By n and m we denote the number of vertices and the number of edges of G, respectively. By $\Delta(G)$ we denote the maximum degree over all vertices of graph G. If $W \subset V(G)$ is a nonempty set then by $G[W]$ we denote a subgraph of G induced by W. The chromatic sum is defined as follows [6].

Definition 1. *By the* ***chromatic sum*** *of graph G we mean $\sum(G) = \min_c \sum(G,c)$, where $\sum(G,c) = \sum_{v\in V(G)} c(v)$ and $c : V(G) \to \mathbb{N}$ is a proper vertex coloring of G, i.e. $c(v) \neq c(w)$ whenever $\{v,w\} \in E(G)$. A coloring c of G is said to be a* ***best coloring*** *if $\sum(G,c) = \sum(G)$.*

The problem of verifying the inequality $\sum(G) \leq k$ for a graph G and arbitrary positive integer k is known as the Chromatic Sum Problem. This differs from the Sum Coloring Problem, which requires a best coloring c in addition. The notion of chromatic sum was first introduced in [6], where the authors showed that the Chromatic Sum Problem is NP-complete on arbitrary graphs. Another complexity result comes from [11], where NP-completeness has been proved for interval graphs. In [2] the authors have shown that there exists $\varepsilon > 0$, such that there is no $(1+\varepsilon)$-ratio approximation algorithm for the Sum Coloring Problem on bipartite graphs, unless P=NP. In [7] the author proved

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the NP-completeness of the Chromatic Sum Problem on cubic planar graphs. Moreover, in [7] the Chromatic Sum Problem on r-regular graphs was proved to be NP-complete for any $r \geq 3$. The 2-approximation algorithm for interval graphs have been shown in [9]. In [1] the authors showed $(\Delta+2)/3$-approximation algorithm for graphs with bounded degree and the 2-approximation algorithm for line graphs.

In this paper we deal with the Chromatic Sum Problem on bipartite graphs. We establish a borderline of intractability for this problem on bipartite graphs with bounded degree, namely the case $\Delta = 3$ is easy, $\Delta = 5$ is hard. We construct an $O(n^2)$-time algorithm for bipartite graphs with $\Delta \leq 3$, and a 27/26-approximation algorithm for the Sum Coloring Problem on any bipartite graph. This improves the previously best known 10/9-approximation algorithm for this problem [2].

1.1 NP-Completeness Results on Bipartite Planar Graphs with $\Delta \leq 5$

In this extended abstract we omit the proofs of NP-completeness.

Theorem 1 ([8]). *The* Chromatic Sum Problem *is NP-complete on planar bipartite graphs with $\Delta \leq 5$.*

Corollary 1 ([8]). *The* Chromatic Sum Problem *is NP-complete on planar bipartite graphs with $\Delta \leq 5$, even when restricted to graphs for which there exists a best 3-coloring.*

2 Exact and Approximation Algorithms

In this section we introduce an idea of 3-pseudocolorings of bipartite graphs and construct an algorithm for finding the best pseudocoloring for any bipartite graph in $O(mn)$ time.

Definition 2. *By a* ***pseudocoloring*** *(3-**pseudocoloring**) of bipartite graph G we mean any mapping $q : V(G) \to \{1, 2, 3\}$ satisfying conditions: every set $C_i := q^{-1}(\{i\})$ is an independent set in graph G for $i = 1$ and $i = 2$. Analogously to Definition 1, by the* ***pseudochromatic sum*** *of a bipartite graph G we mean $\sum_{qs}(G) := \min_q \sum(G, q)$, where $\sum(G, q) := \sum_{v \in V(G)} q(v)$ and q is a pseudocoloring of G. A pseudocoloring q of graph G is said to be a* ***best pseudocoloring*** *if $\sum(G, q) = \sum_{qs}(G)$.*

For any bipartite graph G we have an obvious

Proposition 1. $\sum_{qs}(G) \leq \sum(G) \leq 3n/2$.

We get at once

Proposition 2. *For any best pseudocoloring q of subcubic (i.e. $\Delta(G) \leq 3$) bipartite graph G we have $\Delta(G[C_2 \cup C_3]) \leq 2$.*

Before we show the algorithm, we need some well-known notation of minimum cuts in weighted digraphs (e.g. see [10]). Let $D = (V, A)$ be any digraph without loops and multiple edges, and let w be a vector of positive weights (including ∞) on the edges of D. For any two different vertices $s, t \in V(D)$ by the $s - t$ cut (or simply cut) we mean a partition (S, T) of the set $V(D)$ such that $s \in S$, $t \in T$, $S \cap T = \emptyset$ and $S \cup T = V(D)$. By the capacity of the cut (S, T) we mean $f(S, T) := \sum_{e \in A(D) \cap (S \times T)} w(e)$. By the minimum cut $f_o(D, w, s, t)$ we mean the $s - t$ cut of weighted digraph D which minimizes $f(S, T)$.

Theorem 2. *There exists an algorithm for finding the best pseudocoloring of any bipartite graph in $O(mn)$ time.*

Proof. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. We construct the digraph D with weights w such that a minimum $s - t$ cut (P, Q) for some vertices $s \in P$ and $t \in Q$ is equal to $f_o(D, w, s, t) = f(P, Q) = \sum_{qs}(G) - n(G)$.

Let $G^* = (V_1^* \cup V_2^*, E^*)$ be the isomorphic copy of G such that $V(G) \cap V(G^*) = \emptyset$. By v^* we denote an image of vertex $v \in V(G)$ under isomorphism $h : V(G) \to V(G^*)$, i.e. $h(v) = v^*$ $(h^{-1}(v^*) = v)$, analogously $h(V_i) = V_i^*$. The directed graph D with weights w shown in Figure 1 is formally defined as follows:

$$\begin{aligned}
V(D) &= V(G^*) \cup V(G) \cup \{s\} \cup \{t\} \qquad (1)\\
A(D) &= A_{1,2} \cup A_{2,1} \cup A_{s,1} \cup A_{2,t} \cup A_{1,1} \cup A_{2,2}\\
w(e) &= \begin{cases} 1 & \text{if } e \in A_{s,1} \cup A_{2,t} \cup A_{1,1} \cup A_{2,2} \\ \infty & \text{if } e \in A_{1,2} \cup A_{2,1} \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
A_{1,2} &= \{(v_1, v_2) : v_1 \in V_1 \wedge v_2 \in V_2 \wedge \{v_1, v_2\} \in E(G)\}\\
A_{2,1} &= \{(v_2, v_1) : v_1 \in V_1^* \wedge v_2 \in V_2^* \wedge \{v_1, v_2\} \in E(G^*)\}\\
A_{s,1} &= \{s\} \times V_1\\
A_{2,t} &= V_2 \times \{t\}\\
A_{1,1} &= \{(v_1^*, v_1) : v_1 \in V_1\}\\
A_{2,2} &= \{(v_2, v_2^*) : v_2 \in V_2\}
\end{aligned}$$

Let (P, Q) be the minimum $s - t$ cut in D. We introduce auxiliary notations (see Figure 1) for $i = 1, 2$:

$$\begin{aligned}
P_i = V_i \cap P,\ P_i^* = V_i^* \cap P \qquad (2)\\
Q_i = V_i \cap Q,\ Q_i^* = V_i^* \cap Q,
\end{aligned}$$

moreover, using the isomorphism h we define $P_{1,Q}^* = h(Q_1) \cap P$, $Q_{2,P}^* = h(P_2) \cap Q$. Because $f(P, Q) \leq \sum_{e \in A_{s,1}} w(e) = |V_1| < \infty$, from the infinity of weights of edge sets $A_{1,2}$ and $A_{2,1}$ we get

$$A(D) \cap ((P_2^* \times Q_1^*) \cup (P_1 \times Q_2)) = \emptyset. \qquad (3)$$

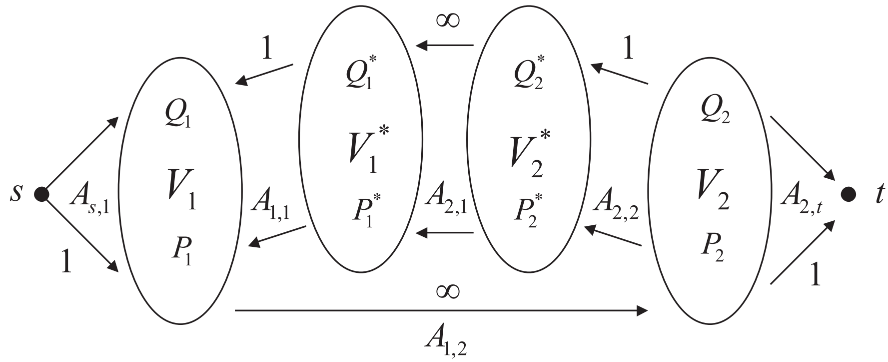

Fig. 1. The directed graph D with specified sets of vertices, edges and its weights.

So, from definition of capacity we obtain $f(P,Q) = |Q_1| + |P_2| + |P^*_{1,Q}| + |Q^*_{2,P}|$. Moreover, if $h(P_1) \cap Q^*_1 \neq \emptyset$ then we can change the (P,Q) partitioning by moving these vertices from Q to P. Observe, that this operation cannot increase the cut capacity and can be done in linear time. Analogously, if $h(Q_2) \cap P^*_2 \neq \emptyset$ the we can move these vertices from P to Q. Therefore in the following we assume that $Q^*_1 \subseteq h(Q_1)$ and $h(Q_2) \subseteq Q^*_2$. So, we have

$$\begin{aligned} f(P,Q) &= |Q_1| + |P_2| + |P^*_{1,Q}| + |Q^*_{2,P}| \\ &= |Q_1| + |P_2| + |h(Q_1)| - |h(Q_1) \cap Q| + |h(P_2)| - |h(P_2) \cap P| \\ &= 2 \cdot |Q_1| + 2 \cdot |P_2| - |Q^*_1| - |P^*_2|. \end{aligned} \tag{4}$$

Now, we shall show the connection between the constructed minimum cut (P,Q) and some pseudocoloring of G. We prove the following claims:
Claim 1. $C_1 := P_1 \cup Q_2$ and $C_2 := h^{-1}(Q^*_1 \cup P^*_2)$ are independent sets in G.
Claim 2. Defining $C_3 := V(G) \backslash (C_1 \cup C_2)$ we get the pseudocoloring q defined as follows: $q^{-1}(\{i\}) = C_i$ with $\sum(G,q) = f(P,Q) + n(G)$.
Claim 3. Pseudocoloring q is the best one, i.e. $\sum(G,q) = \sum_{qs}(G)$.

By (3) $C_1 = P_1 \cup Q_2$ is an independent set in G and $Q^*_1 \cup P^*_2$ is an independent set in G^* and because h^{-1} is an isomorphism so C_2 is an independent set in G. Claim 1 is proved. Then q is a pseudocoloring of G, hence by (4) we get Claim 2:

$$\begin{aligned} \sum(G,q) &= |C_1| + 2 \cdot |C_2| + 3 \cdot |C_3| = 3 \cdot n(G) - 2 \cdot |C_1| - |C_2| \\ &= n(G) + 2 \cdot (n(G) - |C_1|) - |C_2| = n(G) + f(P,Q). \end{aligned}$$

Now, observe that for any pseudocoloring p of G the following partition (S^p, T^p):

$$\begin{aligned} S^p &= \{s\} \cup (C'_1 \cap V_1) \cup h(C'_1 \cap V_1) \cup (V_2 \backslash C'_1) \cup h(V_2 \cap C'_2) \cup h(V_1 \cap C'_3) \\ T^p &= \{t\} \cup (C'_1 \cap V_2) \cup h(C'_1 \cap V_2) \cup (V_1 \backslash C'_1) \cup h(V_1 \cap C'_2) \cup h(V_2 \cap C'_3) \end{aligned}$$

is an $s-t$ cut of capacity $f(S^p, T^p) = \sum(G,p) - n(G)$, where $C'_i := p^{-1}(\{i\})$. Because (P,Q) is the minimum cut in D we get that q is the best pseudocoloring of G, so we have proved Claim 3.

We can construct the minimum cut (P,Q) in $O(mn)$ time using the Ford-Fulkerson algorithm (see [10]), hence we have constructed the best pseudocoloring q of graph G in polynomial time.

As the first consequence of Theorem 2 we get an $O(n^2)$-time algorithm for solving the Sum Coloring Problem on subcubic bipartite graphs.

Theorem 3. *The* Sum Coloring Problem *on subcubic bipartite graphs can be solved in $O(n^2)$ time.*

Proof. Let G be any subcubic bipartite graph. Because $m = O(n)$, so by Theorem 2 we can construct in time $O(n^2)$ the best pseudocoloring q such that every $C_i := q^{-1}(\{i\})$ is an independent set in G for $i = 1, 2$. By Proposition 2 we conclude that the subgraph of G induced by $C_2 \cup C_3$ is of degree at most 2. Because q is the best pseudocoloring of G, we can easily recolor graph $G[C_2 \cup C_3]$ with colors $2, 3$ and get a proper coloring c of graph G using only 3 colors with the same sum of colors. From Proposition 1 it follows $\sum(G,c) = \sum(G,q) = \sum_{qs}(G) \leq \sum(G)$, hence c is the best coloring of G.

In [1] the authors proposed a 9/8-approximation algorithm, which has been improved in [2].

Theorem 4 ([2]). *There exists a* 10/9*-approximation algorithm for the* Sum Coloring Problem *on bipartite graphs.*

Now, we improve on this result by using the pseudocoloring algorithm given in the proof of Theorem 2.

Theorem 5. *There exists a* 27/26*-approximation algorithm for the* Sum Coloring Problem *on bipartite graphs of complexity $O(mn)$.*

Proof. Let $G = (V_1 \cup V_2, E)$ be any bipartite graph with m edges, n vertices and assume that $|V_1| \geq |V_2|$. By Theorem 2 we can construct the best pseudocoloring q in $O(mn)$ time. Let us denote, $C_i := q^{-1}(\{i\})$ and $a_i := |C_i|$ for $i = 1, 2, 3$. Proposition 1 implies

$$\sum(G,q) = a_1 + 2a_2 + 3a_3 = 2n - a_1 + a_3 \leq \sum(G). \tag{5}$$

Now, consider three algorithms A_1, A_2 and A_3 for coloring a bipartite graph G. By A_1 we mean an algorithm that colors V_1 with color 1 and V_2 with color 2. It is easy to see that

$$S(A_1) \leq 3n/2, \tag{6}$$

where by $S(A_i)$ we denote the sum of colors used by algorithm A_i for $i = 1, 2, 3$. The algorithm A_2 colors all the vertices from C_1 with color 1 and colors graph $G[C_2 \cup C_3]$ analogously to A_1 with colors 2 and 3. It is easy to see that

$$S(A_2) \leq a_1 + 5(a_2 + a_3)/2 = 5n/2 - 3a_1/2. \tag{7}$$

Finally, let A_3 be an algorithm that colors C_1 with 1, C_2 with 2 and colors graph $G[C_3]$ similarly to A_1 with colors 3 and 4, hence we get

$$S(A_3) \le a_1 + 2a_2 + 7a_3/2 = 2n - a_1 + 3a_3/2. \tag{8}$$

Now, let A be an algorithm that colors graph G using A_1, A_2, A_3 and chooses the solution with minimum sum of colors. Using 6, 7, 8 and 5 we get

$$\begin{aligned} 26S(A) &\le 2S(A_1) + 6S(A_2) + 18S(A_3) \\ &\le 54n - 27a_1 + 27a_3 = 27(2n - a_1 + a_3) \le 27\sum(G). \end{aligned}$$

In contrast to the general case, where the Chromatic Sum Problem on r-regular graphs is NP-complete [7], the Chromatic Sum Problem on bipartite regular graphs appears to be polynomially solvable. In fact, we get an exact formula for the chromatic sum.

Theorem 6. *The chromatic sum of a connected bipartite regular graph is equal to $3n/2$ for any $n > 1$. Moreover, any coloring c using more than two colors has a greater sum.*

Proof. Consider an arbitrary feasible coloring c of k-regular graph with n vertices. Then

$$k \sum_{v \in V} c(v) = \sum_{\{v,u\} \in E} (c(v) + c(u)) \ge \sum_{\{v,u\} \in E} 3 = 3|E| = 3kn/2,$$

hence $\sum(G) \ge 3n/2$. The lower bound is attained for bipartite regular graphs by coloring with 1 all vertices in one part of the bipartition, and by coloring with 2 all vertices in the other part.

3 Conclusions

The results given in the previuos section tighten the borderline between P and NP-completeness for the Chromatic Sum Problem on low-degree bipartite graphs, namely: graphs with $\Delta \le 3$ are easy instances and those with $\Delta \le 5$ are

Table 1. Complexity classification for the chromatic sum problem on graphs with bounded degree.

Problem: CSP or SCP on graphs	Complexity	Reference
$\Delta \le 2$	$O(n)$	[7]
regular bipartite	$O(n)$	Thm. 6
planar cubic graphs	NPC	[7]
k-regular ($k \ge 3$)	NPC	[7]
bipartite subcubic ($\Delta \le 3$)	$O(n^2)$	Thm. 3
bipartite with $\Delta \le 5$	NPC	Thm. 7

hard. A still open question is the complexity of the problem on bipartite graphs with $\Delta = 4$. The authors conjecture that this problem is polynomially solvable, but this case claim seems to be very hard to prove.

The proposed approximation algorithm produces a coloring that is less than 4% worse than the value of optimal solution. In [2] the authors show that there exists an $\varepsilon > 0$, such that there is no $(1 + \varepsilon)$-ratio approximation algorithm (unless P=NP). We still don't know how far is 1/26 from this ε. Table 1 summarizes the complexity results proved for graphs with small degree.

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4 Appendix

The reduction uses the restriction of the classical NP-complete problem $3DM$ [4], namely a planar $3DM$ problem introduced and proved in [3].

Definition 3. *$3DM_p$: let W, X, Y be three disjoint sets satisfying $|W| = |X| = |Y| = q$ and let M be any subset of $W \times X \times Y$. For every $a \in W \cup X \cup Y$ we define $\#a := |\{(w, x, y) \in M : w = a \vee y = a \vee x = a\}|$ which is equal to 2 or 3. Moreover, a bipartite graph $G = (W \cup X \cup Y \cup M, \{\{a, m\} : a \in m, a \in W \cup X \cup Y, m \in M)$ is planar, where $a \in m$ means that a is one of the coordinates of vector m. The question that we state is as follows: is there a subset $M' \subseteq M$*

satisfying $|M'| = q$, such that every two elements $m_1, m_2 \in M'$, $m_1 \neq m_2$, differ on each coordinate?

The following easy observation holds for any best colorings of graph G.

Proposition 3. *Given a graph G and a decomposition of G into vertex disjoint subgraphs $G_1, ..., G_k$ such that $\bigcup_{i=1}^{k} V(G_i) = V(G)$ and $\bigcup_{i=1}^{k} E(G_i) \subset E(G)$ implies $\sum(G) \geq \sum_{i=1}^{k} \sum(G_i)$. Moreover, if c_i is a best coloring of G_i for all $i = 1, ..., k$ and all these colorings form a coloring c of G, then c is a best coloring of G and $\sum(G) = \sum(G, c) = \sum_{i=1}^{k} \sum(G_i, c_{|V(G_i)})$.*

Theorem 7. *The* Chromatic Sum Problem *is NP-complete on planar bipartite graphs with $\Delta \leq 5$.*

Proof. We show a polynomial reduction from problem $3DM_p$ to the Chromatic Sum Problem on planar bipartite graphs with degree bounded by 5. This reduction is a modification of NP-completeness proof of the CSP for subcubic planar graphs showed in [7]. Let W, X, Y, q, M be given as in Definition 3 and let x_i be the number of elements $a \in W \cup X \cup Y$ such that $\#a = i$ ($i = 2$ or $i = 3$).

We define a graph G as follows

$$V(G) = \{v_m : m \in M\} \cup \bigcup_{a \in W \cup X \cup Y} V\left(A^a_{\#a}\right) \tag{9}$$

$$E(G) = \{\{a_m, v_m\}, \{b_m, v_m\}, \{c_m, v_m\} : m = (a, b, c) \in M\} \cup \bigcup_{a \in W \cup X \cup Y} E\left(A^a_{\#a}\right),$$

where $a \in W \cup X \cup Y$ and A^a_2 ($\#a = 2$) or A^a_3 ($\#a = 3$) are bipartite graphs with the desired properties of the best colorings.

First, we construct an auxiliary bipartite graph B with non-symmetry property of every best coloring. Consider the bipartite graph B with $\Delta = 5$ shown in Figure 2.

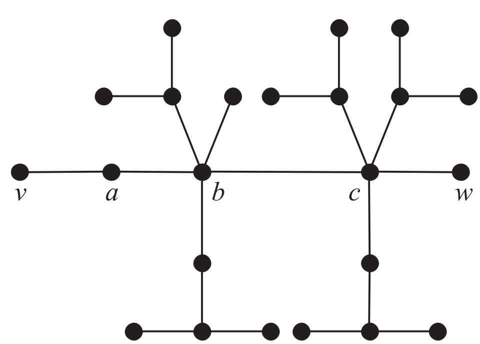

Fig. 2. The auxiliary graph B with the chromatic sum 33.

We will show the following property: for every best coloring c of B vertex v is colored with 2 and w is colored with 1. Moreover, if we color the pair of vertices (v, w) with a pair of colors $(1, 2)$ we can extend this partial colorig to the coloring of graph B in such a way that the sum of colors exceeds the chromatic sum of B exactly by 1.

By T_b we denote a connected subgraph of $B \backslash \{a, c\}$ including vertex b, analogously by T_c we mean that tree of $B \backslash \{b, w\}$ including vertex c. Let $T_v = B[\{v, a\}]$ and $T_w = B[\{w\}]$. For a given graph G and a vertex $v' \in V(G)$ let $bc\,(G, v') = \{k \in \mathbb{N} : c\,(v') = k \wedge c$ any best coloring of $G\}$ be a list of colors. Analogously, for any $v', w' \in V(G)$ let $bc(G, (v', w')) = \{(k, l) : c\,(v') = k \wedge c\,(w') = l \wedge c$ any best coloring of $G\}$. Analyzing all the best colorings of the defined trees we get $bc\,(T_v, v) = \{1, 2\}$, $bc\,(T_v, a) = \{1, 2\}$, $bc\,(T_w, w) = \{1\}$, $bc\,(T_b, b) = \{2, 3\}$ and $bc(T_c, c) = \{1, 3\}$. Moreover $\sum(T_v) = 3$, $\sum(T_w) = 1$, $\sum(T_b) = 13$ and $\sum(T_c) = 16$, hence by Proposition 3 we obtain $\sum(G) = 33$ and there is only one coloring c_p of the path $B[\{v, a, b, c, w\}]$ that can be extended to best coloring of the whole graph B, namely c_p colors the vertices v, a, b, c, w with colors $2, 1, 2, 3, 1$, respectively. Now, color vertex v with 1 and w with 2. Coloring the vertex a with 2, b with 3 and c with 1 we can extend this pre-coloring to the whole graph B with the sum of colors equal to 34.

We construct a graph A_2^a shown in Figure 3 for a given element $a \in W \cup X \cup Y$ occuring only in $x, y \in M$.

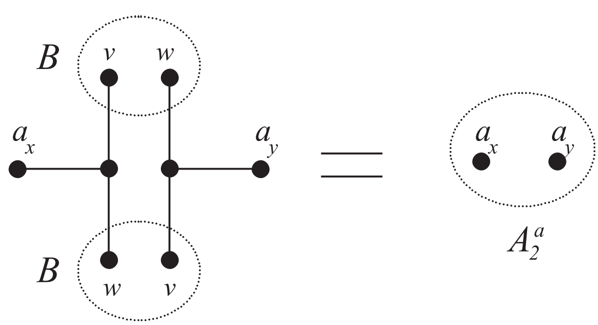

Fig. 3. Graph A_2^a with the chromatic sum 73.

Notice, that graph A_2^a is bipartite with $\Delta = 5$ and $bc(A_2^a, a_x) = \{1, 2\}$, $bc(A_2^a, a_y) = \{1, 2\}$. By Proposition 3 we have $\sum(A_2^a) \geq 2 \cdot 33 + 6 = 72$, from the properties of B it follows $\sum(A_2^a) > 72$. On the other hand, one can easily construct colorings of A_2^a with the chromatic sum equal to 73. Considering all possibilities of coloring of the vertices a_x and a_y we get at once $bc(A_2^a, (a_x, a_y)) = \{(1, 2), (2, 1)\}$.

At last, for a given element $a \in W \cup X \cup Y$ occuring only in $x, y, z \in M$ we construct a graph A_3^a shown in Figure 4.

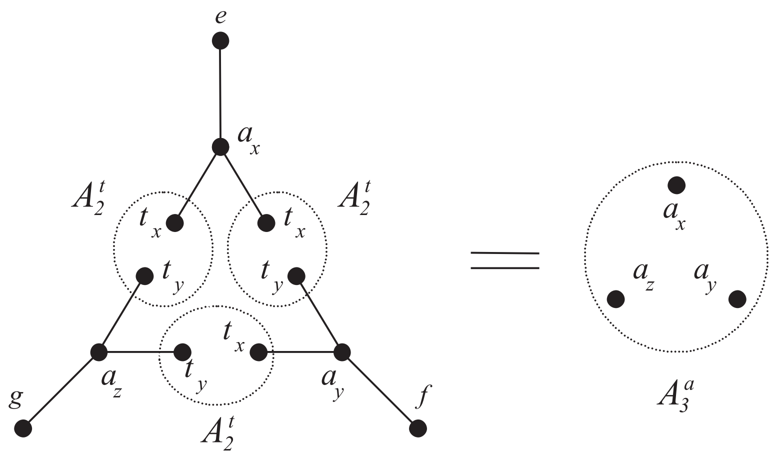

Fig. 4. Bipartite graph A_3^a with the chromatic sum 229.

Notice, that graph A_2^t is just an auxiliary graph. First, let us note that graph A_3^a is bipartite with $\Delta \leq 5$ and by Proposition 3 we have $\sum(A_3^a) \geq 3 \cdot 73 + 3 \cdot 3 = 228$, but it is impossible to extend best colorings of all A_2^t-graphs to the whole graph A_3^a, so $\sum(A_3^a) > 228$. On the other hand, one can easily construct a coloring with the sum equal to 229. Let c be any best coloring of A_3^a, i.e. $\sum(A_3^a, c) = 229$. There are only two possibilities:

(1) $c(\{a_x, a_y, a_z\}) = \{1, 2, 3\}$ and the coloring c restricted to any A_2^t-graph is the best coloring, or

(2) $c(\{a_x, a_y, a_z\}) = \{1, 2\}$ and only one A_2^t-graph is colored with sum greater than 73.

In both cases $\{1, 2\} \subset c(\{a_x, a_y, a_z\})$. Moreover, coloring any vertex from set $\{a_x, a_y, a_z\}$ with 1 and the others with 2 we can extend this pre-coloring to the best coloring of A_3^a.

Now we are able to show that there exists a proper solution M' to $3DM_p$ if and only if there exists a coloring c satisfying $\sum(G, c) \leq k$, where $k = 73 \cdot x_2 + 229 \cdot x_3 + 2 \cdot q + (|M| - q)$. Let us notice that the graph defined in (9) is bipartite with $\Delta(G) \leq 5$ and by Definition 3 it is planar.

Now, suppose that M' is a proper solution of $3DM_p$. We define a coloring c as folows: $c(v_m) = 2$ if $m \in M'$ and $c(v_m) = 1$ if $m \in M \backslash M'$. For any $a \in W \cup X \cup Y$ we color the graphs $A_{\#a}^a$with 3 colors such that $c(a_m) = 1$, whenever $m \in M'$ and $c(a_m) = 2$, if $m \notin M'$. Based on the properties of graphs $A_{\#a}^a$ we can extend the coloring c to the whole graph G so that $\sum(G, c) \leq k$.

Conversely, suppose that c is a coloring of the graph G satisfying $\sum(G, c) \leq k$. Now let $\sum_M := \sum_{m \in M} c(v_m) > |M| + q$. We conclude that

$$\sum_{a \in W \cup X \cup Y} \sum(A_{\#a}^a) = \sum(G, c) - \sum_M < 73 \cdot x_2 + 229 \cdot x_3,$$

which is impossible. Thus suppose that exactly $p < q$ vertices among all $|M|$ vertices v_m are colored with a color different from 1. Hence at most $3 \cdot p$ graphs $A^a_{\#a}$ have neighbors in set $\{v_m : m \in M \wedge c(v_m) \geq 2\}$ and at least $3 \cdot (q-p)$ graphs $A^a_{\#a}$ are colored with $2, 3, \ldots$. This gives

$$\begin{aligned}\sum(G,c) &= \sum_{a \in W \cup X \cup Y} \sum\left(A^a_{\#a}\right) + \sum\nolimits_M \geq \\ &\geq 73 \cdot x_2 + 229 \cdot x_3 + 3 \cdot (q-p) + |M| + p > k\end{aligned}$$

which is impossible. Hence $\sum_M = |M| + q$ and exactly q vertices v_m are colored with 2, we get the desired equality $\sum(G,c) = k$. Thus we have constructed the solution $M' = \{m \in M : c(v_m) = 2\}$ in polynomial time.

Note that simply replacing graphs A^a_2 by edges $\{a_x, a_y\}$ and similarly A^a_3 by triangles we can prove NP-completeness for planar subcubic graphs [7].