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Consecutive colorings of the edges of general graphs

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Abstract

Given an n-vertex graph G, an edge-coloring of G with natural numbers is a *consecutive* (or interval) *coloring* if the colors of edges incident with each vertex are distinct and form an interval of integers. In this paper we prove that if G has a consecutive coloring and $n \geqslant 3$ then $S(G) \leqslant 2n-4$, where $S(G)$ is the maximum number of colors allowing a consecutive coloring. Next, we investigate the so-called deficiency of G, a natural measure of how far it falls of being consecutively colorable. Informally, we define the deficiency $\operatorname{def}(G)$ of G as the minimum number of pendant edges which would need to be attached in order that the resulting supergraph has such a coloring, and compute this number in the case of cycles, wheels and complete graphs. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider a relatively new concept of graph coloring, namely a consecutive edge-coloring problem. Given a proper coloring of the edges of G with colors $1, 2, 3, \ldots$ the coloring is said to be a *consecutive coloring* if for each vertex the colors of the edges incident form an interval of integers. This model of coloring has immediate applications in scheduling theory, in the case when an optimal schedule without waiting periods or idle times is searched for. In this application the problem can be modeled as a graph whose vertices correspond to processors, edges represent unit execution time biprocessor tasks and colors correspond to assigned time units [9]. A no-wait production environment typically arises from characteristics of the processing

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technology (e.g. temperature, viscosity) or from the absence of storage capacity between tasks of a job (e.g. lack of buffers).

This particular variation of edge coloring apparently was first studied under the name of 'interval coloring' by Asratian and Kamalian [2] and by Sevastjanov [11]. However, their papers were devoted to bipartite graphs only. For example, Sevastjanov [11] proved that it is NP-complete to decide if a given bipartite graph G admits a consecutive coloring of edges. Recently, Giaro [3] has strengthened this result by showing that the problem of deciding consecutive Δ-colorability of such a graph is easy if $\Delta \leqslant 4$ and becomes NP-complete for $\Delta \geqslant 5$, where Δ is the maximum vertex degree of G.

In the present paper we consider the consecutive edge-coloring problem for general graphs. Since not all graphs have consecutive colorings, we introduce a new graph invariant: the consecutive edge-coloring deficiency, and discuss its properties in general and in some special cases. The rest of this paper is organized as follows. In Section 2 we give basic definitions and facts concerning the consecutive edge-coloring problem. In particular, we give a simple proof that the general consecutive coloring problem is NP-complete. Section 3 is devoted to graphs allowing consecutive coloring of edges. The main result of this section is that the maximal number of colors used for an n-vertex graph G with $n > 2$ is bounded by $2n - 4$, which is a slight improvement over the bound $2n-1$ given in [2]. We also show that this bound is very close to tight. In addition, we show that in contrast to Vizing's bound the minimum number of colors required for G to be consecutively colorable is not bounded in terms of Δ. Section 4 is devoted to graphs that do not allow a consecutive coloring of edges. Any such graph can be augmented to a consecutively colorable supergraph by attaching some pendant edges to its vertices. The minimum number of edges whose attachment to G makes that such a supergraph has a consecutive coloring is just the *deficiency* of G. In particular, we establish the deficiency of the following graphs: odd cycles, wheels, broken wheels, and complete graphs.

2. Basic definitions and results

All graphs considered in this paper are simple and connected. Given such a graph $G=(V,E)$, by $n(G)$ and $\Delta(G)$ we denote the number of vertices and maximum vertex degree in G, respectively. We shall drop the reference to the graph, writing n and Δ, if G is clear from context. $\boldsymbol{N}$ denotes the set of positive integers.

Definition 2.1. Given a finite subset A of $\boldsymbol{N}$, by the *deficiency* $\operatorname{def}(A)$ of A we mean the number of integers between $\min A$ and $\max A$ not belonging to A.

Clearly, $\operatorname{def}(A) = \max A - \min A - |A| + 1$. A set A with $\operatorname{def}(A) = 0$ is an *interval*.

Definition 2.2. Given a graph G and its proper edge-coloring $c : E(G) \to \boldsymbol{N}$, by the deficiency of *$c$ at vertex v*, $\operatorname{def}(G, c, v)$, we mean the deficiency of the set of colors of

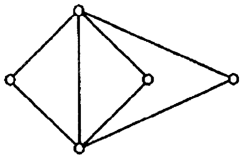

Fig. 1. Graph $K_{1,1,3}$.

edges incident with $v \in V(G)$. The *deficiency of coloring* c is the sum of deficiencies of all vertices in G denoted $\mathrm{def}(G,c) = \sum_{v \in V} \mathrm{def}(G,c,v)$. The *deficiency of graph* G, $\mathrm{def}(G)$, is the minimum $\mathrm{def}(G,c)$ among all possible colorings c of G. A graph G with deficiency d is called *d-deficient*, and a coloring c for which $\mathrm{def}(G,c) = 0$ is called *consecutive*.

Not all graphs admit a consecutive coloring of edges. The smallest counter-example is K_3.

Let $\chi'(G)$ denote the chromatic index of G. Suppose G is 0-deficient and let c denote any consecutive coloring of its edges. We define sets $E_i = \{e \in E(G)\colon c(e) \equiv i \,(\mathrm{mod}\, \Delta)\}$, $i = 1, \ldots, \Delta$. It is easy to see that each set E_i is a matching in G. Therefore coloring the edges of E_i with color i for each i gives a Δ-coloring of G. Thus we have

Proposition 2.3. *If G is consecutively colorable, then $\chi'(G) = \Delta$.*

By well-known Vizing's theorem [12] $\chi'(G) \leqslant \Delta + 1$, and thus we have only two possibilities for the chromatic index of a graph. Accordingly, G is said to be of *Class* 1 if $\chi'(G) = \Delta$, and of *Class* 2 if $\chi'(G) = \Delta + 1$. In this paper 0-deficient graphs will be called graphs of *Class* 0. Therefore, by Proposition 2.3, graphs of Class 0 belong to Class 1. However, the converse is not true since there are Class 1 graphs that are not consecutively colorable. The smallest such example is shown in Fig. 1.

It was proved for regular graphs that verification whether $\chi'(G) = \Delta$ is NP-complete [10]. On the other hand, given a regular graph G, this graph is Δ-colorable iff G is of Class 0, since any Δ-coloring of such a graph is consecutive (an example of such a graph is K_{2n}). This immediately implies the following well-known proposition.

Proposition 2.4. *Deciding if G is of Class* 0 *is NP-complete.*

Definition 2.5. Let G be a graph of Class 0. By the span $s(G,c)$ of coloring c we mean the number of colors used in the consecutive coloring. By $s(G)$ ($S(G)$) we denote the *minimum* (*maximum*) *span* among all consecutive colorings of G, respectively.

Clearly, for any Class 0 graph G

$$\Delta(G) \leqslant s(G) \leqslant S(G).$$

Besides regular graphs of Class 1, simple examples of 0-deficient graphs are trees, complete bipartite graphs, doubly convex bipartite graphs [1], grid graphs [4], bipartite

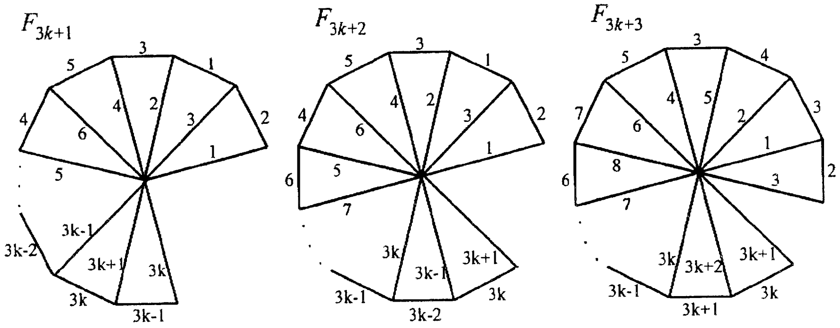

Fig. 2. A consecutive coloring of F_n.

outerplanar graphs [13], bipartite cacti [5] as well as bipartite $(2,\Delta)$-regular graphs [7]. A consecutive coloring of fans F_n, $n>3$ is shown in Fig. 2. It is easy to see that if the *core* of G, i.e. the subgraph obtained by successive pruning away all vertices of degree 1, is of Class 0, then so is G. The converse is not true since any odd cycle with a pendant edge attached to it is consecutively colorable.

In [8] the following property of trees and complete bipartite graphs has been shown: *for any* $k\in\{s(G),\ldots,S(G)\}$ *there is a consecutive coloring* c *such that* $s(G,c)=k$. This property does not hold for general bipartite graphs. Sevastjanov [11] gave an example of a bipartite graph of Class 0 whose span takes on the values of 100 and 173 only.

3. Graphs which are consecutively colorable

In this section we give tight bounds on $S(G)$, if G is a graph of Class 0.

Let c be a consecutive coloring of G. By $c_{\min}$ and $c_{\max}$ we denote the minimum and maximum color used in c, respectively. Obviously, every integer between $c_{\min}$ and $c_{\max}$ is a color of some edge. Let $v_1,\ldots,v_m$ be a sequence of vertices of any path joining an edge colored with $c_{\min}$ and an edge colored with $c_{\max}$. From the consecutiveness of c it follows that the difference of colors assigned to any two edges incident with $v\in V(G)$ is at most $\deg(v)-1$. Thus

$$S(G,c)=c_{\max}-c_{\min}+1\leqslant 1+\sum_{i=1}^{m}(\deg(v_i)-1). \tag{3.1}$$

In this way we obtain two simple bounds on the maximum span.

Proposition 3.1. *For any graph G of Class* 0

$$S(G)\leqslant 1+\max_{P\in\boldsymbol{P}}\sum_{i=1}^{n(P)}(\deg(v_i^P)-1), \tag{3.2}$$

where $\boldsymbol{P}$ *is the set of all simple paths of G.*

Proposition 3.2. *For any graph G of Class* 0

$$S(G) \leqslant (\mathrm{diam}(G)+1)(\Delta(G)-1)+1, \tag{3.3}$$

where $\mathrm{diam}(G)$ *is the diameter of* G.

The bound of Proposition 3.2 was improved in the case of bipartite graphs by Asratian and Kamalian [2]. They took advantage of simple fact that in any 0-deficient graph there are two vertices of edges with $c_{\max}$ and $c_{\min}$ connected by a path of length $\leqslant \mathrm{diam}(G)-1$. Thus by inequality (3.1) we have

Proposition 3.3. *For any bipartite graph G of Class* 0

$$S(G) \leqslant \mathrm{diam}(G)(\Delta(G)-1)+1. \tag{3.4}$$

Note that the bound (3.2) is tight for trees and the bound (3.4) is tight for graphs $K_{m,m}$.

3.1. New bound

Now we shall find an upper bound on $S(G)$ in terms of n only. Let c be a consecutive coloring of graph G and let $c_{\min} > 1$. We augment G to G' by introducing two new vertices and joining one of them by edge $\underline{e}$ with an endvertex of any edge colored $c_{\min}$ and the other vertex by edge $\bar{e}$ with an endvertex of any edge colored $c_{\max}$. Next we expand the coloring c by giving color $c_{\min}-1$ to $\underline{e}$ and color $c_{\max}+1$ to $\bar{e}$. In this way we obtain a coloring c' of G' in which precisely two edges $\underline{e}$ and $\bar{e}$ get extreme colors.

In order to prove our bound $S(G) \leqslant 2n-4$ we need some additional notions. By an *e-path* in G' we mean any simple path $(e_0,\ldots,e_m)$, $m \geqslant 1$ such that $e_0 = \underline{e}$ and $e_m = \bar{e}$. Given an e-path $p=(e_0,\ldots,e_m)$ by an *i-vertex* of p we mean the only vertex $v_i \in V(G)$ belonging to both e_i and e_{i+1}, $0 \leqslant i < m$, and by an *i-hair* $(0 \leqslant i < m)$ of p we mean any edge $e \in E(G)$ incident with an i-vertex whose color $c(e)$ is strictly between $c(e_i)$ and $c(e_{i+1})$. Finally, by an *i-node* we mean the endvertex of an i-hair other than its i-vertex. Thus any i-hair has two endpoints: one called its i-vertex and the other called its i-node. Also, no i-hair of $p=(e_0,\ldots,e_m)$ belongs to p.

By the *skeleton* of e-path p we mean a subgraph of G generated by the edges $e_1,\ldots,e_{m-1}$ and all i-hairs of p. Clearly, every skeleton is connected and all its vertices are i-vertices or i-nodes for $i=0,\ldots,m-1$. Moreover, for each $k \in \{c_{\min},\ldots,c_{\max}\}$ there is a skeleton edge e such that $k=c(e)$. Therefore, $S(G,c)$ is less than or equal to the number of edges in some skeleton.

Definition 3.4. Let P be the set of all e-paths in G'. We define a function $\Psi : P \to \boldsymbol{N}^3$ assigning to each $p=(e_0,\ldots,e_m)$ a triple of integers $\Psi(p)=(l(p), w(p), \mathrm{pp}(p))$, where:

- $l(p)$ is the length of p

$$l(p) = m + 1,$$

- $w(p)$ is the width of p

$$w(p) = 1 + \sum_{i=0}^{m-1} |c(e_i) - c(e_{i+1})|,$$

- $\mathrm{pp}(p)$ is the power parameter of p

$$\mathrm{pp}(p) = \sum_{i=0}^{m} (2^{c(\bar{e})} - 2^{c(e_i)}).$$

An e-path p is called *minimal* if Ψ attains the minimum value for it with respect to the natural lexicographic order in $\boldsymbol{N}^3$.

Now we are in the position to prove an important lemma.

Lemma 3.5. *For any minimal e-path $p = (e_0, \ldots, e_m)$ in G' the following holds*:

1. *No i-vertex is j-node for any j $(0 \leqslant i, j < m)$.*
2. *If there exists i-node which is also j-node $i \neq j$*, then $|i - j| = 1$.

Proof. Both statements will be proved by the way of contradiction according to the following scheme: the negation of the thesis implies the existence of an e-path p' fulfilling $\Psi(p') < \Psi(p)$, with respect to the lexicographic order in $\boldsymbol{N}^3$, which contradicts that p is minimal.

1. Let i-vertex v also be a j-node for some $i < j$ (the case $i > j$ is similar). Then $j > i + 1$. Let e be the j-hair joining v with j-vertex. Note that $p' = (e_0, \ldots, e_i, e, e_{j+1}, \ldots, e_m)$ is an e-path and $l(p') < l(p)$, so $\Psi(p') < \Psi(p)$.
2. Let i-node v be also a j-node for some $i < j$ and let $|i - j| > 1$. Suppose e' is the i-hair joining v with i-vertex and e'' is the j-hair joining v with j-vertex. Then $p' = (e_0, \ldots, e_i, e', e'', e_{j+1}, \ldots, e_m)$ is an e-path. If $|i - j| > 2$ then $l(p') < l(p)$ and $\Psi(p') < \Psi(p)$. Thus assume $|i - j| = 2$. Now we have $l(p') = l(p)$. If $c(e_i), \ldots, c(e_{i+3})$ does not form a monotone sequence then $w(p') < w(p)$, and consequently $\Psi(p') < \Psi(p)$. So suppose $c(e_i) < c(e_{i+1}) < c(e_{i+2}) < c(e_{i+3})$ (the other case is similar). Then $w(p') = w(p)$, but $\mathrm{pp}(p') < \mathrm{pp}(p)$. In fact, since $c(e') < c(e_{i+1})$ and $c(e_{i+2}) < c(e'')$, so

$$\begin{aligned} \mathrm{pp}(p) - \mathrm{pp}(p') &= -2^{c(e_{i+1})} + 2^{c(e')} - 2^{c(e_{i+2})} + 2^{c(e'')} \\ &> 2^{c(e'')} - 2^{c(e_{i+2})} - 2^{c(e_{i+1})} > 2^{c(e'')} - 2 \times 2^{c(e_{i+2})} \\ &= 2^{c(e'')} - 2^{c(e_{i+2})+1} \geqslant 0. \end{aligned}$$

Thus $\Psi(p') < \Psi(p)$, a contradiction. □

We are now in a position to state our main result.

Theorem 3.6. *If graph G is of Class* 0 *with $n \geqslant 3$ vertices, then*

$$S(G) \leqslant 2n - 4. \tag{3.5}$$

Proof. Let c be a consecutive coloring of $G \neq K_2$ and H be the skeleton of any minimal e-path $(e_0, \ldots, e_m)$. Let n' and n'' stand for the number of all vertices and the number of i-nodes in H, respectively. Of course, $n' = n'' + m$. By Lemma 3.5 the degree of any i-node in H is 1 or 2. Therefore,

$$S(G,c) \leqslant |E(H)| \leqslant 2n'' + m - 1 = 2n' - (m+1) \leqslant 2n - (m+1). \tag{3.6}$$

Hence Theorem 3.6 holds provided that H contains at least three i-vertices. If H contains only one i-vertex, then all i-nodes are pendant in H and H is a star. Thus $S(G,c) \leqslant |E(H)| = n' - 1 \leqslant n - 1 \leqslant 2n - 4$.

It remains to consider the case $m = 2$. Assuming that $S(G,c) \geqslant 2n - 3$ we have to assume that all inequalities in (3.6) become equalities. Consequently, all colors on the edges of H are different, each i-node has degree 2 in H, $V(G) = V(H)$, and $n'' > 0$. From the above it follows that the colors of n'' 1-hairs constitute the interval $\{c(e_1) - n'', \ldots, c(e_1) - 1\}$. Similarly, the colors of n'' 2-hairs constitute the interval $\{c(e_1) + 1, \ldots, c(e_1) + n''\}$. Let s be the sum of the number of non-skeleton edges of G over all 1-nodes. It is easy to see that $2s \geqslant \sum_{j=1}^{n''}(c(e_1) + j) - \sum_{j=1}^{n''}(c(e_1) - j) - \sum_{j=1}^{n''} 1 = n''^2$. On the other hand, the number of such edges is not greater than $n''(n''-1)/2$. This contradiction proves the theorem. $\square$

Theorem 3.7. *Let G be a triangle-free graph of Class* 0. *Then*

$$S(G) \leqslant n - 1. \tag{3.7}$$

Proof. Let G be a graph without triangles. Let c be a consecutive coloring of G and H its skeleton as in the previous theorem. Then the number of vertices in H is $n' \leqslant n$. Since G is triangle-free, so is G', and no vertex in H is an i-node and $(i+1)$-node simultaneously. Thus by Lemma 3.5, H is a tree and, $S(G,c) \leqslant n' - 1 \leqslant n - 1$. $\square$

Corollary 3.8. *For any bipartite graph G of Class* 0 *we have $S(G) \leqslant n - 1$.*

How tight are these bounds? As far as Theorem 3.7 is concerned it is easy to see that any path P_n can be colored consecutively with colors $1, 2, \ldots, n-1$, so $S(P_n) = n - 1$. The same holds true for complete bipartite graphs. Hypercubes Q_m, however, form a class of bipartite graphs for which bound (3.3) is generally better than (3.7) because they are low-diameter. Now, consider a sequence of graphs $G_m = \bigcup_{i=1}^{2m-1}(K_{2m}^i + K_{2m}^{i+1})$, where $\cup$ and $+$ is union and join of graphs, respectively. We color the edges of K_{2m}^1 with colors $1, \ldots, 2m-1$, the edges of $K_{2m}^1 + K_{2m}^2$ incident to any vertice $u \in K_{2m}^1$ with colors $2m, \ldots, 4m-1$, the edges of K_{2m}^2 with colors $4m, \ldots, 6m-2$, and so forth. Proceeding in this way we obtain

$$S(G_m) \geqslant (2m-1)(4m-1) + 2m - 1 = 8m^2 - 4m.$$

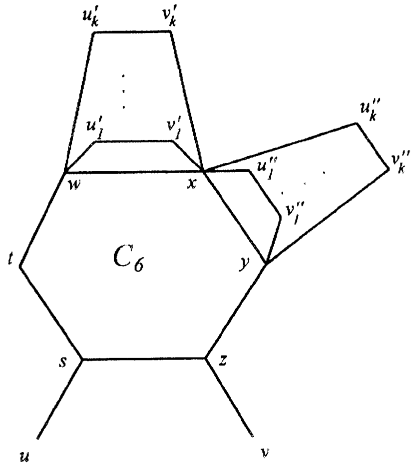

Fig. 3. Graph $G_{1,k}$.

Since $n(G_m) = 4m^2$, so $S(G_m) \geqslant 2n - 2\sqrt{n}$ and $\lim_{m\to\infty} S(G_m)/n(G_m) = 2$. Thus the coefficient 2 in (3.5) cannot be improved. Moreover, for some $n \in N$ there are n-vertex graphs having consecutive colorings with strictly more than $2n - 2\sqrt{n}$ colors. In fact, we found an algorithm for consecutive edge coloring of K_{2^k} using $2 \times 2^k - k - 2$ colors for every $k \in \boldsymbol{N}$.

3.2. Construction

In contrast to the chromatic index of G, the minimum span is not bounded in terms of Δ. To show this we shall give a sequence of consecutively colorable graphs $G_{m,k}$ such that $\Delta(G_{m,k}) = 2k + 2$ and $s(G_{m,k}) \geqslant 2m - 1$, for $k \geqslant 15$. Graph $G_{1,k}$ is shown in Fig. 3 and graph $G_{m,k}$ $(m \geqslant 2)$ is constructed inductively.

At the edge $\{w,x\}$ we build k paths $w \to u_i' \to v_i' \to x$ and similarly k paths on $\{x,y\}$. The vertex of maximum degree is x and $\Delta(G_{1,k}) = 2k + 2$. Edges $\{u_i', v_i'\}$ and $\{u_i'', v_i''\}$ will be called *special* and edges $\{u,s\}$ and $\{v,z\}$ will be called *terminal*. Graph $G_{m+1,k}$ is constructed from $G_{1,k}$ by replacing every special edge with graph $G_{m,k}$, so that for each $i = 1,\dots,k$ vertices u_i', v_i' and u_i'', v_i'' are identified with vertices u, v of the corresponding $G_{m,k}$. Note that $G_{1,k}$ can be colored consecutively in such a way that $\{u,s\}$ and $\{v,z\}$ have the same color, say a. In order to do this the edges of central cycle C_6 are colored alternately with $a+1$ and $a+2$, the edges $\{w,u_i'\}$ and $\{x,v_i'\}$ are colored with $a+2+i$ and $\{u_i', v_i'\}$ with colors $a+1+i$. Finally, the edges $\{x,u_i''\}$ and $\{y,v_i''\}$ get colors $a+1-i$ and edges $\{u_i'', v_i''\}$ - colors $a+2-i$. By immediate induction all graphs $G_{m,k}$ can be colored in this way. Thus all graphs $G_{m,k}$ are of Class 0. It remains to estimate the value of $s(G_{m,k})$. First, observe that in any consecutive coloring of $G_{m,k}$ the colors of edges $\{u,s\}$ and $\{v,z\}$ can differ by at most 4. Let c be any consecutive coloring of $G_{m,k}$, $m > 1$, $k \geqslant 15$. Let $c(vz) = c(\{v,z\}) = a$ and $c(us) = c(\{u,s\}) = b$. We shall show that among $4k$ terminal edges of $G_{m-1,k}$ there is one with color less than $\min\{a,b\}$. Suppose this is not true and let $a \leqslant b$. Then all colors of $2k$ edges $\{x,v_i'\}$ and $\{x,u_i''\}$ must be at least $a-1$. On the other hand, the consecutiveness of c implies

$$c(yz) \leqslant a+2, \quad c(yv_i'') \leqslant a+k+3, \quad c(xu_i'') \leqslant a+k+9,$$
$$c(tw) \leqslant a+5, \quad c(wu_i') \leqslant a+k+6, \quad c(xv_i') \leqslant a+k+12.$$

Thus considering the last inequality we conclude that $2k$ edges incident at vertex x have at their disposal at most $k+14 < 2k$ colors, a contradiction. Now, suppose $b < a$. Then the lower bound on the colors of edges $\{x,v_i'\}$ and $\{x,u_i''\}$ is $b-1$ and

$$c(yz) \leqslant b+4, \quad c(yv_i'') \leqslant b+k+5, \quad c(xu_i'') \leqslant b+k+11,$$
$$c(tw) \leqslant b+3, \quad c(wu_i') \leqslant b+k+4, \quad c(xv_i') \leqslant b+k+10.$$

Hence our $2k$ edges on vertex x have at most $k+13$ colors, a contradiction. Similarly, a certain edge among $4k$ terminal edges of $2k$ graphs $G_{m-1,k}$ obtains a color greater than $\max\{a,b\}$, otherwise by multiplying all colors by -1 we would get a coloring contradicting the previous result. Therefore, some terminal edge of certain $G_{m-1,k}$ replacing a special edge in $G_{m,k}$ obtains in c a color $\leqslant a-1$, some terminal edge of $G_{m-2,k}$, which replaces in this $G_{m-1,k}$ a special edge has in c a color $\leqslant a-2$, etc. Finally, a certain terminal edge of some $G_{1,k}$ has a color $\leqslant a-(m-1)$. Similarly, a certain edge of some $G_{1,k}$ is assigned a color $\geqslant a+(m-1)$ in c. Thus $s(G_{m,k}) \geqslant 2m-1$. Therefore for $k=15$ $\lim_{m\to\infty} s(G_{m,15})/\Delta(G_{m,15}) \geqslant \lim_{m\to\infty} (2m-1)/32 = \infty$.

4. Graphs which are not consecutively colorable

Before we establish the deficiency of some families of graphs, we give a simple interpretation of this notion. Let G be a graph with $\operatorname{def}(G) > 0$ and let c be a coloring realizing the deficiency $\operatorname{def}(G)$. We construct a graph G' by introducing a new vertex v' and joining it to a vertex $v \in V(G)$ for which $\operatorname{def}(G,c,v) > 0$. Next we define a coloring c' of G' such that $c'(e) = c(e)$ for each $e \in E(G)$ and $c'(v,v') = m$, where m is a missing color at v in c. It is easy to see that $\operatorname{def}(G',c') = \operatorname{def}(G) - 1$. On the other hand, adjoining to a graph any pendant edge, i.e. an edge attached to an existing vertex whose second endpoint is a new vertex of degree 1, cannot decrease its deficiency by more than 1, so $\operatorname{def}(G') = \operatorname{def}(G) - 1$. Thus we have

Remark 1. The deficiency of graph G is equal to the minimum number of pendant edges whose attachment to G makes a graph consecutively colorable.

The deficiency of G seems to be a new graph invariant. In this section we will be able to evaluate it for a few special families of graphs.

Theorem 4.1. *Let G be a Δ-regular graph with odd number of vertices. Then*

$$\operatorname{def}(G) \geqslant \frac{\Delta}{2}. \tag{4.1}$$

Proof. Of course, $\Delta(G)$ is even. We label the vertices of G with numbers 1 through n. Assume c is a coloring of G with $\operatorname{def}(G,c) < \Delta/2$. Also let x_j^i be ith color (increasingly) at vertex v_j and n_j^i be the quantity of l's such that $i \equiv x_j^l \pmod{\Delta}$.

Notice that if $n_j^i > 2$ for some i,j then there would be colors x_j^a and x_j^b such that $x_j^a - x_j^b \geqslant 2\Delta$. In this case the deficiency at v_j would be

$$\operatorname{def}(G,c,v_j) \geqslant 2\Delta - 1 - (\Delta - 2) = \Delta + 1 > \Delta.$$

So suppose that $n_j^i \in \{0,1,2\}$. Let

$$z_j = |\{i\colon n_j^i = 2\}|,$$
$$y_j = |\{i\colon n_j^i \equiv 0 \pmod 2\}|.$$

Since $\sum_{i=0}^{\Delta-1} n_j^i = \Delta$, so $y_j = 2z_j$ and if $z_j > 0$, then at least z_j colors at v_j exceed $x_j^1 + \Delta - 1$. Thus $x_j^\Delta \geqslant x_j^1 + \Delta - 1 + z_j$. Consequently, $\operatorname{def}(G,c,v_j) = x_j^\Delta - x_j^1 - \Delta + 1 \geqslant z_j$ and

$$y_j \leqslant 2\operatorname{def}(G,c,v_j).$$

For any $i \in \{0,\ldots,\Delta - 1\}$, by the handshaking lemma, $\sum_{j=1}^n n_j^i \equiv 0 \pmod 2$. Since n is odd so there is j such that $n_j^i \equiv 0 \pmod 2$ and thus $\sum_{j=1}^n y_j \geqslant \Delta$. Now $\Delta \leqslant \sum_{j=1}^n 2\operatorname{def}(G,c,v_j) = 2\operatorname{def}(G,c)$, a contradiction. □

A simple example of graphs for which bound (4.1) is tight are odd cycles. In fact, by Theorem 4.1 their deficiency is at least 1, and by coloring the first edge of C_{2k+1} with 1 and the remaining edges with 2 and 3 alternately, we obtain a coloring c with $\operatorname{def}(C_{2k+1},c) = 1$. Thus

$$\operatorname{def}(C_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 4.2.

$$\operatorname{def}(K_n) = \begin{cases} 0 & \textit{if } n \textit{ is even}, \\ \lfloor n/2 \rfloor & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. If $n = 2k$, $k \in \boldsymbol{N}$ then K_n is 0-deficient as mentioned in Section 2. Thus $\operatorname{def}(K_{2k}) = 0$. If $n = 2k + 1$ then $\Delta = n - 1$ and $\lfloor n/2 \rfloor = \Delta/2 = k$. By Theorem 4.1, $\operatorname{def}(K_{2k+1}) \geqslant k$. Now it suffices to show a coloring c of K_{2k+1} with the deficiency of c equal to k. We distinguish two cases.

(i) *k is odd*. The subgraph induced by vertices $1,2,\ldots,k+1$ is isomorphic to K_{k+1} which is k-regular and can be consecutively colored with $1,\ldots,k$ colors at each vertex. Similarly, the subgraph induced by vertices $1, k+2, k+2, \ldots, 2k+1$ can be consecutively colored using colors $2k+1,\ldots,3k$ at each vertex. The remaining edges form a complete bipartite graph $K_{k,k}$ which can be colored with colors $k+1,\ldots,2k$ at each vertex. Consequently, each vertex $2,\ldots,k+1$ meets colors of $\{1,\ldots,2k\}$, each vertex $k+2,\ldots,$ $2k+1$ meets colors of $\{k+1,\ldots,3k\}$, and vertex 1 has all its edges colored with $\{1,\ldots,k\} \cup \{2k+1,\ldots,3k\}$. Thus $\operatorname{def}(K_{2k+1},c) = k$.

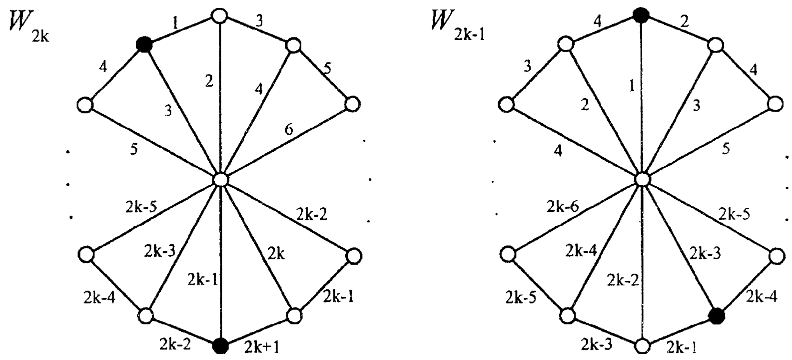

Fig. 4. A coloring of W_n, $n \geqslant 13$.

(ii) *k is even*. Let $i, j \in \{1, \dots, k\}$. First, we color consecutively the edges of subgraph K_{2k} on vertices $2, \dots, 2k+1$. For each i, j we color the edges joining vertices $i+1$ and $j+k+1$ (they create a complete bipartite graph $K_{k,k}$) by assigning each such edge color $i+j$. In this way the edges at vertices $i+1$ and $i+k+1$ get colors from $\{i+1, \dots, i+k\}$. Next, all colors of value greater than $k+1$ are increased by $k-1$. Now vertices $i+1$ and $i+k+1$ meet colors $\{i+1, \dots, i+2k-1\} - \{k+2, \dots, 2k\}$. The only uncolored edges of K_{2k} are those within the subgraph on vertices $2, \dots, k+1$ and the subgraph on vertices $k+2, \dots, 2k+1$. They induce two separated graphs K_k which are consecutively colorable with colors $k+2, \dots, 2k$ at each vertex. In this way we obtain a consecutive coloring of K_{2k} with colors $\{i+1, \dots, i+2k-1\}$ at vertices $i+1$ and $i+k+1$, for all i. Finally, we color the edges joining vertices 1 and $i+1$ with color i and edges joining vertices 1 and $i+k+1$ with color $i+2k$. Therefore, the deficiency at vertices $2, \dots, 2k+1$ is 0. Since vertex 1 meets colors of $\{1, \dots, k\} \cup \{2k+1, \dots, 3k\}$, so $\operatorname{def}(K_{2k+1}, c) = k$. $\square$

The next class of graphs whose deficiency is known are wheels and wheels without one edge. We recall that wheel W_n is a join of C_{n-1} and K_1.

Theorem 4.3. *The deficiency of wheel W_n is*

$$\operatorname{def}(W_n) = \begin{cases} 0 & \text{if } n = 4, 7, 10, \\ 1 & \text{if } n = 3, 5, 6, 8, 9, 11, 12, \\ 2 & \text{if } n \geqslant 13. \end{cases}$$

Proof. We first show that $\operatorname{def}(W_n) \leqslant 2$. A coloring c for which $\operatorname{def}(W_n, c) = 2$ is given in Fig. 4.

The reader can check by inspection that $\operatorname{def}(W_n) = 0$ for $n = 4, 7, 10$ and $\operatorname{def}(W_n) = 1$ for $n = 3, 5, 6, 8, 9, 11, 12$. To find out whether $\operatorname{def}(W_n, c) \leqslant 1$, we wrote a computer program. Results of computer computations showed that wheels W_n are not 1-deficient, if $n \geqslant 13$. $\square$

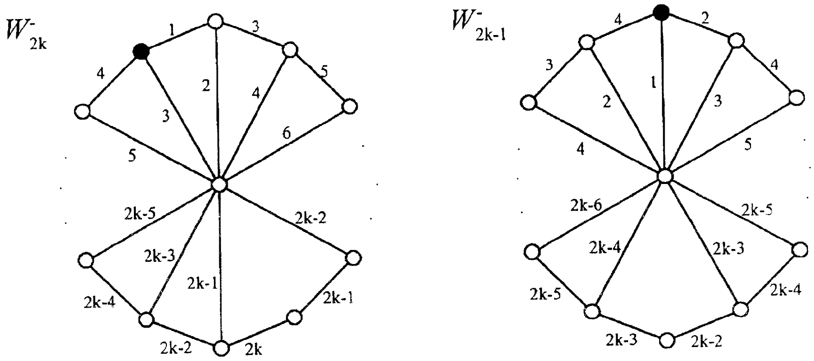

Fig. 5. Colorings of W_n^-, $n \geqslant 11$.

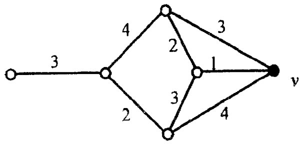

Fig. 6. Kite K.

A *broken wheel* is a wheel without one spoke. In the following, a broken wheel with n vertices is denoted by W_n^-.

Theorem 4.4.

$$\operatorname{def}(W_n^-) = \begin{cases} 0 & \textit{if } n = 3,4,6,7,9,10, \\ 1 & \textit{if } n = 5,8 \textit{ and } n \geqslant 11. \end{cases}$$

Proof. The colorings proving that $\operatorname{def}(W_n^-) \leqslant 1$ are shown in Fig. 5. The reader can check that $\operatorname{def}(W_n^-) = 0$ if $n = 3,4,6,7,9,10$. As above, we verified computationally that these are the only broken wheels of Class 0. □

We conclude our investigation of deficiency of graphs with two asymptotic results. We first show that $\operatorname{def}(G)$ is not bounded in terms of $\Delta(G)$. To this aim let us consider the graph of Fig. 6, called a *kite* K. The coloring depicted in this figure and the fact that K is of Class 2 imply that $\operatorname{def}(K) = 1$. We construct a sequence of graphs H_m with $\Delta(H_m) = 3$ as follows. H_m consists of a path P_m, each vertex of which is attached to a kite as shown in Fig. 7. It is easy to see that for each $m \in N$, $\operatorname{def}(H_m) = m$. Thus $\lim_{m \to \infty} \operatorname{def}(H_m)/\Delta(H_m) = \infty$.

Similarly as for the span of a graph, the deficiency of G is bounded in terms of n. It would be interesting to find a tight upper bound on $\operatorname{def}(G)$. Clearly, by Vizing's theorem [12] the edges of G can be colored with at most $\Delta + 1$ colors, which implies that for any $v \in V(G)$, $\operatorname{def}(G,c,v) \leqslant \Delta - 1$. Thus $\operatorname{def}(G) \leqslant n(\Delta - 1) < (n-1)^2$. What is the value of $\operatorname{def}(G)$ in the worst case? A partial result in this direction is given

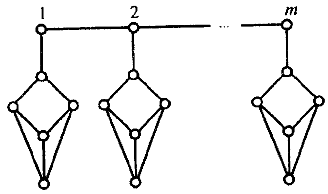

Fig. 7. Graph H_m.

in [6]. The authors found a family of graphs G_m with m^2+m+2 vertices for which $\mathrm{def}(G_m)=m^2-3m-2$. Thus $\lim_{m\to\infty}\mathrm{def}(G_m)/n(G_m)=1$.

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