Equitable 4-coloring of cacti and edge-cacti in polynomial time

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Abstract. A graph is equitably \( k \)-colorable if its vertices can be partitioned into \( k \) independent sets in such a way that the number of vertices in any two sets differ by at most one. In the paper we establish the smallest such \( k \), denoted \( \chi_e(G) \), for cacti and edge-cacti without \( K_2 \) and pendant \( K_2 \). In addition, we give a polynomial-time algorithm for optimal equitable coloring of such graphs. In this way we give a new class of planar graphs for which the problem remains polynomially solvable.

1 Introduction

An assignment of colors to the vertices of graph \( G \), one color to each vertex, is called an equitable coloring, if adjacent vertices are assigned different colors and the sizes of the color classes differ by at most one. The smallest integer \( k \) for which \( G \) is equitably \( k \)-colorable is called the equitable chromatic number of \( G \) and denoted by \( \chi_e(G) \). In the following any equitable coloring of \( G \) with \( \chi_e(G) \) colors will be called optimal. From the fact that equitable coloring is a legal coloring of vertices, we have \( \chi(G) \leq \chi_e(G) \). Applications of equitable coloring can be found in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. As we know, we can model this problem as coloring the vertices of a graph \( G \) whose nodes correspond to classes, edges correspond to time conflicts between classes and colors to hours. If the set of available rooms is strongly restricted then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. Recently Penman and Hefling bounds. If we can equitably color graph with constant number of colors, then we can improve such bounds.

The notion of equitable colorability was introduced by Meyer [6]. However, an earlier work of Hajnal and Szemerédi [3] showed that a graph \( G \) with maximum degree \( \Delta \) is equitably \( k \)-colorable if \( k \geq \Delta(G) + 1 \). In 1973, Meyer [6] formulated the following conjecture:
**Equitable Coloring Conjecture (ECC) [6]:** For any connected graph \( G \), other than complete graph or odd cycle, \( \chi^-(G) \leq \Delta(G) \).

We also have a stronger conjecture:

**Equitable \( \Delta \)-Coloring Conjecture [1]:** Let \( G \) be a connected graph with maximum degree \( \Delta \). Suppose \( G \not\in \{K_n,C_{2n+1},K_2n+1,2n+1\} \) for any \( n \geq 1 \). Then \( G \) is equitably \( \Delta \)-colorable.

The Equitable \( \Delta \)-Coloring Conjecture was proved for some classes of graphs, e.g. bipartite graphs [5], outerplanar graphs with \( \Delta \geq 3 \) [8] and planar graphs with \( \Delta \geq 13 \) [9].

The general problem of deciding if \( \chi^-(G) \leq 3 \) is \( NP \)-complete. If, however, \( G \) is a tree then its vertices can be equitably colored to optimality in \( O(n\Delta) \) time [1]. Except for trivial cases like complete bipartite graphs, wheels, hypercubes, complete graphs and cycles, Chen-Lih’s algorithm has been known as the only polynomial-time algorithm for optimal equitable coloring. In this paper we consider a more general class of graphs, namely cacti and edge-cacti (also known as polygon trees). Since we are interested in using as few colors as possible, we assume that our graphs have neither \( K_3 \) nor pendant \( K_2 \). In this way we obtain a class of planar graphs which are equitably 4-colorable at worst. We give a polynomial-time algorithm for optimal equitable coloring of such graphs. In addition, we show that if \( G \) belongs to our class of graphs and \( \Delta \geq 5 \), then \( \chi^+(G) \leq \lfloor \Delta/2 \rfloor + 1 \), where \( \chi^+(G) \) is the least \( k \) for which \( G \) has an equitable \( k \)-coloring for every \( k' \geq k \). Thus we prove that Yap-Zhang’s conjecture [8] holds for the graphs under consideration.

## 2 Definition

First, we define families of graphs considered in this paper. A pendant vertex is a vertex with degree 1, and a pendant edge is any edge incident with such a vertex.

**Definition 1.** Trees are the smallest family of graphs including single vertex and closed under operation of attaching a new pendant edge to a vertex.

By analogy to the above recursive definition of trees we have the following three definitions.

**Definition 2.** Edge-cacti are the smallest family of graphs including all cycles and closed under operation of attaching a new cycle to a single edge, i.e. identifying this edge with some edge of the attached cycle.

**Definition 3.** Cacti are the smallest family of graphs including single vertex and closed under operation of attaching a new pendant edge or cycle to a vertex.

Now we can generalize our graph classes to so-called thorny graphs (see Fig.1), as follows:
Definition 4. Thorny graphs are the smallest family of graphs including single vertex and closed under the following operations:

- attaching a pendant edge to a vertex,
- attaching a cycle to a vertex,
- attaching a cycle to an edge.

![diagram]

Fig. 1. An example of: (a) cactus; (b) edge-cactus; (c) thorny graph.

Corollary 1 ([2]). We have the following:

- every cactus is a thorny graph,
- every connected outerplanar graph is a thorny graph,
- every edge-cactus is a thorny graph,
- every thorny graph is a partial 2-tree,
- every thorny graph is connected, planar and tripartite.

Note that trees can have arbitrarily high equitable chromatic number.

Proposition 1. For stars we have $\chi_e(K_{1,n}) = \lceil n/2 \rceil + 1$.

Corollary 2. Determining if a given connected graph is equitably 2-colorable can be done in linear time.

Proof. Clearly, $\chi_e(G) = 1$ iff $G = K_1$, otherwise $\chi_e(G) = 2$ is true only if $G$ is bipartite and the cardinalities of partitions differ by at most one. We can determine this in $O(n + m)$ time.
3 Main results

Definition 5. For a given coloring of graph $G$ with colors $\{1, \ldots, k\}$ the counter of this coloring is a sequence $(n_1, \ldots, n_k)$, where $n_i$ is the number of vertices colored $i$, for $i = 1, \ldots, k$. A counter is equitable if $\max_i |n_i - n_j| \leq 1$.

Thus, for any counter $(n_1, \ldots, n_k)$ we have $n_1 + \cdots + n_k = n = |V(G)|$.

Lemma 1. If $G$ can be equitably colored with $k$ colors, $k \geq 3$, then $G'$, obtained by inserting $l \geq 2$ new vertices into any edge of $G$, can be equitably colored with $k$ colors.

Proof. If $l = 2$ then a coloring for $G'$ can be obtained by simple expanding the equitable coloring of $G$. In Fig.2 we can see a modified edge and its colors.

![Fig. 2. Inserting 2 vertices to an edge.](image)

Thus, irrespectively of colors assigned to the vertices of that edge, we can assign any pair of different colors to new vertices. It is easy to see that for any equitable counter of length $k$ we can choose different numbers $i$ and $j$, such that increasing $n_i$ and $n_j$ by 1 gives an equitable counter. The same holds if $l = 3$, as shown in Fig. 3, inserting $l > 3$ vertices can be seen as a result of series of the above operations.

Lemma 2. If $G$ can be equitably colored with $k$ colors, $k \geq 3$, then $G'$ obtained by attaching a new cycle $C_{l+2}$, $l > 1$ to an edge of $G$ can be equitably colored with $k$ colors.

Proof. The above operation induces to the insertion of $l$ new vertices into an edge and joining its end vertices by a new edge.

It is easy to see that there is no constant bound for the equitable chromatic number of cacti which include stars. It seems that these are the pendant vertices that cause unlimited growth of $\chi_e(G)$. However, among bridgeless cacti there
Fig. 3. Inserting 3 vertices.

are other cases causing troubles, namely triangles. Let $S^t_k$ be a graph consisting of $k$ cycles $C_2$ sharing single common vertex. If $G = S^t_k$, then $\chi_s(G) > k$, since it is a supergraph of star $K_{1,2k}$. In a similar way we can construct bridgeless bipartite cacti, which have $\chi_s(G) > 2$ ($S^4_2$ is a bipartite graph with cardinality of partitions equal to 4 and 6) and bridgeless cacti without triangles with $\chi_s(G) > 3$. For example, consider $S^8_3$ with the central vertex $v$. We assume that there is an equitable 3-coloring of $S^8_3$. Suppose vertex $v$ is colored with 1. Thus $n_2 + n_3 \leq 2(n_1 + 1)$, since $4k + 1 = n \leq 3n_1 + 2$, but among five vertices of each cycle at most two can be colored with 1, namely: vertex $v$ and another one. Thus $n_1 \leq k + 1$ and $4k + 1 \leq 3k + 5$, which is not true for all natural $k$.

**Theorem 1.** Any thorny graph without pendant vertices and cycles of length 3 or 5 can be equitably colored with 3 colors.

**Proof.** Cycles $C_3$ and $C_4$ can be equitably colored with three colors in an obvious way. Longer cycles can be obtained from $C_3$ or $C_4$ by inserting some new vertices into an edge. First of all, we will systematically extend the current equitable coloring to new cycles. If we attach cycle $C_4$ to a vertex colored with $a$, then we give different colors $b$, $a$ and $c$ consecutively to its vertices. We can treat any cycle longer than $C_5$ as $C_4$ with new vertices inserted into its edges like in Lemma 1. If we attach a cycle $C_{i+2}$, $i > 1$, to an edge we have the situation from Lemma 2. Thus we have proved the thesis for bridgeless thorny graphs.

By contradiction, let $G$ be the smallest thorny graph satisfying the conditions of the theorem and such that it cannot be equitably colored with 3 colors. The above implies that $G$ must include a bridge. Let us consider the last pendant edge added in the construction which is a bridge in $G$. There is at least one cycle attached to its end $v$ ($G$ does not have pendant vertices). Let $D$ be the largest bridgeless subgraph of $G$, including $v$, and let $u$ be the closest to $v$ vertex of degree at least 3 such that $u \in V(G - D)$. It is easy to see that $D$ is a bridgeless thorny graph, $v \in V(D)$ and $G$ is obtained by joining a thorny graph $H, u \in V(H)$, by a path connecting $v$ and $u$. By the hypothesis, $H$ can be equitably colored with 3 colors.

We can restrict the length of the path to 1 or 2, since any longer path can be obtained by putting at least two new vertices on an edge. Similarly, $v \in D$
is an element of a cycle $C$, so we can assume that $D = C_4$. Otherwise, we can reconstruct firstly $C$ and then $D$ by extending the suitable equitable coloring (in a similar way to the part of proof concerning bridgeless thorny graphs). So, there is equitable coloring of $H$ and we extend it as follows: first we attach to $u$ a path of two or three edges whose vertices get colors in such a way that the new coloring is equitable and after that we attach to its pendant edge a new cycle $C_4$ and extend its coloring as previously.

**Theorem 2.** Any thorny graph without pendant vertices and cycles $C_3$ can be equitably colored with $k$ colors for every $k \geq 4$.

**Proof.** The proof is analogous to that of Theorem 1. If $G$ is a cycle then it suffices to make an equitable coloring of $C_3$ and $C_4$ and use Lemma 1. In the case of creating bridgeless thorny graphs by attaching $C_3$ or $C_5$ to a vertex, we give distinct colors to the remaining vertices so that our coloring is still equitable. The case of longer cycles is treated as previously. If we attach a cycle $C_{l+2}$, $l > 1$ to an edge of equitably $k$-colorable graph, then we can use Lemma 2.
The above argument proves the case of bridgeless thorny graphs. Now, let us consider the smallest contrxample $G$, including equitably colored graph with vertex $u$, to which the cycle of the length 4 or 5 was attached by the path of length 1 or 2. We proceed in a similar way to that in the proof of Theorem 1 by attaching to vertex $u$ a path of length 2 or 3, respectively, whose new vertices get distinct colors. After that we attach to its pendant edge a cycle $C_4$ or $C_5$ colored as previously.

Theorems 1 and 2 lead to some simple propositions.

**Corollary 3.** Any edge-cactus without $C_3$ can be equitably colored with $k \geq 3$ colors. Furthermore, if edge-cactus $G$ is bipartite then $\chi_e(G) = 2$.

**Proof.** The first part of the thesis follows from the proof of Theorem 1 and 2. It is easy to see that in the construction of edge-cactus we attach cycle to an edge, so a cycle $C_5$ does not prevent equitable 3-coloring. To prove the second part assume that $G = G(V_1, V_2)$ stands for a bipartite graph. Now it suffices to note that the cardinalities of both $V_1$ and $V_2$ must be the same. This is clear if $G$ is a cycle. By adding a new cycle $C_{2l+2}$ to an edge we add exactly $l$ new vertices to each $V_1, V_2$.

If we drop the condition of being $C_3$-free then the graphs under consideration can have arbitrarily high equitable chromatic number. In fact, $(l + 1)$-vertex fan $F_{l+1}$ is an edge-cacti and $\chi_e(F_{l+1}) > \frac{1}{2}$, because $F_{l+1}$ is a supergraph of a star $K_{1, l}$ with the same set of vertices.

![Fig.6. A fan $F_{l+1}$.](image)

**Corollary 4.** We have the following:

- any cactus without pendant vertices and cycles of length 3 or 5 can be equitably colored with $k$ colors for any $k \geq 3$. 
any bipartite cactus without pendant vertices can be equitably colored with $k$ colors for any $k \geq 3$.

- any triangle-free cactus without pendant vertices can be equitably colored with $k$ colors for every $k \geq 4$.

In [8] Yap and Zhang posed the following question,

**Question 1.** Is it true that if $G$ is an outerplanar graph of degree $\Delta \geq 3$, then $\chi_\pm(G) \leq \lfloor \Delta/2 \rfloor + 1$?

Above theorems and propositions state that answer to this question is positive not only for $\{C_3, C_5, \text{pendant } K_2\}$-free and $\{C_3, \text{pendant } K_2\}$-free with $\Delta > 4$ outerplanar graphs but for $C_3$-free polygon trees as well. Recently, Kostochka proved in [4] that the answer to above question is positive in general case.

4 The algorithm

**Theorem 3.** There exists a polynomial-time algorithm for optimal equitable coloring of a thorny graph without pendant vertices and cycles $C_3$.

**Proof.** The case $\chi_\pm(G) \leq 2$ is obvious, so assume that $\chi_\pm(G) \geq 3$. Although the proof of Theorem 2 has not been presented in a constructive way, it gives a polynomial algorithm, which colors graphs equitably with four colors. If we want to color equitably such a thorny graph $G$, we have to reconstruct its construction from the elementary components (in time $O(n^2)$) and unless $G$ is a cycle, we have to create a smaller graph $G'$. Introducing to $G'$ two or three vertices, attaching cycle $C_4$ or any other operation as described in the proof of Theorem 2 gives $G$. Then we call a recursive procedure which gives equitable coloring of $G'$. Once we have got it we extend the coloring to $G$. The depth of the recursion does not exceed the number of edges, so this part of the algorithm can be done in $O(n^3)$ time.

Now we only need to describe a procedure which checks if a given thorny graph satisfies $\chi_\pm(G) = 3$ and produces optimal coloring, if it is the case. The second part is easy to supplement, so we left it to the reader. We will present the first part only (polynomial verification of the equality).

The idea is as follows: we consider all 3-colorings of $G$ of order $n$ and we check if there is any equitable coloring among them. Because the number of colorings is exponential, we consider only a coloring counter. Thus, we will count sets of counters for every legal coloring with colors $\{1, 2, 3\}$ for subgraphs of $G$ satisfying some additional conditions. It remains to be checked whether there is any equitable counter. Let us note that counters for a graph $H$ satisfy $n_1 + n_2 + n_3 = n(H)$, where $n(H)$ denotes the number of vertices in graph $H$, so their cardinality does not exceed $n^2(H)$ and sets of counters can be easily represented by $(n(H) + 1)$-square array of bits ($[i, j] = 1$, if $(i, j, n(H) - i - j)$ is an element of the set).

Let us introduce some auxiliary operations. For any sets of counters $X$ and $Y$:

$$X + Y = \{(x_1 + y_1, x_2 + y_2, x_3 + y_3) : (x_1, x_2, x_3) \in X \land (y_1, y_2, y_3) \in Y\},$$
\[ X - Y = \{(x_1 - y_1, x_2 - y_2, x_3 - y_3) : (x_1, x_2, x_3) \in X \land (y_1, y_2, y_3) \in Y\}, \]

where \(0 + X = \emptyset\) for any set \(X\). For any permutation \(\pi\) of the set \(\{1, 2, 3\}\) we have:

\[ \pi(X) = \{(x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, x_{\pi^{-1}(3)}) : (x_1, x_2, x_3) \in X\}. \]

It is easy to see that the computational complexity of these operations does not exceed \(O(n^2)\) and \(O(n^3)\), respectively. Lastly, let \(3_i\) be a counter \((n_1, n_2, n_3)\) satisfying \(n_i = 1\) and \(n_j = 0\) for \(j \neq i, i, j = 1, 2, 3\).

The procedure will be based on two functions, mutually calling each other \((H\) denotes any thorny graph, which is a subgraph of \(G\)).

- \(ColV(H, v)^{(i)}\) - a set of counters of every legal coloring of \(H\) with \(\{1, 2, 3\}\), in which a vertex \(v\) is colored with \(i\) (it is important that \(v \in V(H)\)).
- \(ColE(H, v_1, v_2)^{(i)}\) - a set of counters of every legal coloring of \(H\) with \(\{1, 2, 3\}\), in which a vertex \(v_1\) is colored with \(i\) and a vertex \(v_2\) is colored with \(j\) (it is necessary that \(\{v_1, v_2\} \in E(H)\)). Of course \(ColE(H, v_1, v_2)^{(i)} = \emptyset\) if \(i \neq j\).

It is easy to see that we can use the formula \(ColV(H, v)^{(i)} = \pi(ColV(H, v)^{(1)})\) for any permutation satisfying \(\pi(1) = i\). Similarly, \(ColE(H, v_1, v_2)^{(ij)} = \pi(ColE(H, v_1, v_2)^{(12)})\), \(i \neq j\), for the permutation such that \(\pi(1) = i\), \(\pi(2) = j\).

Our goal is to determine \(ColV(G, u)^{(1)}\) for any vertex \(u\). Our recursion is as follows.

- For the function \(ColV(H, v)^{(i)}\):
  - If \(H\) is a one vertex, then \(ColV(H, v)^{(i)} = \{3_1\}\).
  - Otherwise, we find a vertex \(u\) such that \(\{u, v\} \in E(H)\) and count:

\[ ColV(H, v)^{(1)} = ColE(H, v, u)^{(12)} \cup ColE(H, v, u)^{(13)}. \]

- For the function \(ColE(H, v, u)^{(ij)}\):
  - If \(\{u, v\}\) is a bridge in \(H\), we remove it. Now we have two components \(H_1\) and \(H_2\) such that \(v \in V(H_1), u \in V(H_2)\):

\[ ColE(H, v, u)^{(12)} = ColV(H_1, v)^{(1)} + ColV(H_2, u)^{(2)} \]

  - Otherwise, there is the shortest cycle \(C_k = (v_1 = v, v_2 = u, v_3, \ldots, v_k, v_{k+1} = v)\) and the maximal edge-cactus \(H^*\) containing \(C_k\) such that \(H^* \subset H\).

  We can partition \(H^*\) into edge disjoint edge-cacti:

\[ H^* = H_1^* \cup H_2^* \cup \ldots \cup H_k^*, \]

where \(H_i^*, i = 1, 2, \ldots, k\) is such a subgraph that only the edge \(\{v_i, v_{i+1}\}\) from the cycle \(C_k\) belongs to \(H_i^*\). Let

\[ H = H_1 \cup H_2 \cup \ldots \cup H_k, \]

where \(H_i, i = 1, 2, \ldots, k\) is connected edge disjoint subgraph of \(H\) and \(H_i^* \subset H_i\). Now we count sets \(ColE(H_i, v_i, v_{i+1})^{(12)}\) and denote by \(S_{ij}^2\)
a set of counters for coloring of $H_2 \cup \ldots \cup H_{x-1}$ such that $v_2$ is colored with $i$ and $v_x$ with $j$ for $x = 3, \ldots, k + 1$.

Obviously:

$$S_3^{(ij)} = \text{ColE}(H_2, v_2, v_3)^{(ij)}$$

and for the next indices $x$ we have

$$S_x^{(ij)} = \bigcup_{l \in \{1, 2, 3\} \setminus \{j\}} (S_{x-1}^{(il)} + \text{ColE}(H_{x-1}, v_{x-1}, v_x)^{(ij)} - \{3_l\}).$$

Finally,

$$\text{ColE}(H, v, u)^{(12)} = S_{k+1}^{(21)} + \text{ColE}(H_1, v, u)^{(12)} - \{3_1\} - \{3_2\}.$$

With every invocation of the function $\text{ColE}()$ of the recurrence we consider a graph partitioned into edge disjoint subgraphs, so the number of leaves in the recursion tree is linear with respect to the number of edges in the graph. The equality $|V(H)| = O(|E(H)|)$ holds for every connected planar graph. Thus, for a thorny graph, which is planar, this number is $O(n)$.

Therefore the complexity of our procedure does not exceed $O(n^5)$. In fact, we can implement our algorithm in such way that the timing is bounded by $O(n^4)$. We can use Lemma 3. We will describe this technique in a short way. First, we note that the functions $\text{ColE}()$, $\text{ColV}()$ are not important for algorithm. We use them for clarity of the description. In faster implementation we have to use only one recursive function which gives a set of counters of some legal 3-colorings for a subgraph $H \subseteq G$. If we call this function for a graph with at least two edges it should call itself for two edge disjoint subgraphs $H_1$ and $H_2$ of $H$. Let us note that computing inside a single invocation needs at most $O(|V(H_1)||V(H_2)|^2)$ steps and the whole tree of recursive calling is a binary tree with at most $|E(G)|$ leaves. So using Lemma 3 for $l = 2$ we obtain $O(n^4)$ as the complexity of our algorithm.

**Lemma 3.** Let $T$ be a directed binary tree with root $r$ - every vertex not leaf $v$ has two immediate successors: $\text{left}(v) \neq \text{right}(v)$. Furthermore, let $T_v$ be a maximal subtree $T$ with root $v$ (in a particular case $T = T_v$) and let $\text{leaves}(T_v)$ be a set of all leaves in the subtree, for $v \in V(T)$. Then, we have the following assessment for every positive number $l$:

$$\sum_{v \in V(T) \setminus \text{leaves}(T)} (|\text{leaves}(T_{\text{left}(v)})| |\text{leaves}(T_{\text{right}(v)})|)^l \leq |\text{leaves}(T)|^{2l}$$

**Proof.** We have:

$$\sum_{v \in V(T) \setminus \text{leaves}(T)} (|\text{leaves}(T_{\text{left}(v)})| |\text{leaves}(T_{\text{right}(v)})|)^l =$$

$$\sum_{v \in V(T) \setminus \text{leaves}(T)} (|\text{leaves}(T_{\text{left}(v)}|^l \times |\text{leaves}(T_{\text{right}(v)}|^l),$$
but for every such $v$ holds $\text{leaves}(T_{\text{left}(v)})^I \times \text{leaves}(T_{\text{right}(v)})^I \subseteq \text{leaves}(T)^{2I}$ and every such Cartesian products are distinct for distinct vertices $v$ (two vertices $u \in \text{leaves}(T_{\text{left}(v)})$ and $w \in \text{leaves}(T_{\text{right}(v)})$ describe $v$ synonymously). So the last sum does not exceed the cardinality of a set $\text{leaves}(T)^{2I}$.

References