On some Ramsey and Turán-type numbers for paths and cycles

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Abstract

For given graphs \( G_1, G_2, \ldots, G_k \), where \( k \geq 2 \), the multicolor Ramsey number \( R(G_1, G_2, \ldots, G_k) \) is the smallest integer \( n \) such that if we arbitrarily color the edges of the complete graph on \( n \) vertices with \( k \) colors, there is always a monochromatic copy of \( G_i \) colored with \( i \), for some \( 1 \leq i \leq k \). Let \( P_k \) (resp. \( C_k \)) be the path (resp. cycle) on \( k \) vertices. In the paper we show that \( R(P_3, C_k, C_k) = R(C_k, C_k, C_k) = 2k - 1 \) for odd \( k \). In addition, we provide the exact values for Ramsey numbers \( R(P_4, P_4, C_k) = k + 2 \) and \( R(P_3, P_5, C_k) = k + 1 \).

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let \( G \) be such a graph. The vertex set of \( G \) is denoted by \( V(G) \), the edge set of \( G \) by \( E(G) \), and the number of edges in \( G \) by \( e(G) \). \( C_m \) denotes the cycle of length \( m \) and \( P_m \) – the path on \( m \) vertices. For given graphs \( G_1, G_2, \ldots, G_k, k \geq 2 \), the multicolor Ramsey number \( R(G_1, G_2, \ldots, G_k) \) is the smallest integer \( n \) such that if we arbitrarily color the edges of the complete graph of order \( n \) with \( k \) colors, then it always contains a monochromatic copy of \( G_i \) colored with \( i \), for some \( 1 \leq i \leq k \). We only
consider 3-color Ramsey numbers \( R(G_1, G_2, G_3) \) (in other words we color the edges of \( K_n \) with colors red, blue and green). The Turán number \( T(n, G) \) is the maximum number of edges in any \( n \)-vertex graph which does not contain a subgraph isomorphic to \( G \). By \( T'(n, G) \) we denote the maximum number of edges in any \( n \)-vertex non-bipartite graph which does not contain a subgraph isomorphic to \( G \). A non-bipartite graph on \( n \) vertices is said to be extremal with respect to \( G \) if it does not contain a subgraph isomorphic to \( G \) and has exactly \( T'(n, G) \) edges. By \( T^+(n, G) \) we denote the maximum number of edges in any \( n \)-vertex non-bipartite graph which does not contain a subgraph isomorphic to \( G \). For any \( v \in V(G) \), by \( r(v) \), \( b(v) \) and \( g(v) \) we denote the number of red, blue and green edges incident to \( v \), respectively. The degree of vertex \( v \) will be denoted by \( d(v) \) and the minimum degree of a vertex of \( G \) by \( \delta(G) \). The open neighbourhood of vertex \( v \) is \( N(v) = \{ u \in V(G) | \{u, v\} \in E(G) \} \). \( G_1 \cup G_2 \) denotes the graph which consists of two disconnected subgraphs \( G_1 \) and \( G_2 \). \( kG \) stands for the graph consisting of \( k \) disconnected subgraphs \( G \). We will use \( G_1 + G_2 \) to denote the join of \( G_1 \) and \( G_2 \), defined as \( G_1 \cup G_2 \) together with all edges between \( G_1 \) and \( G_2 \).

The remainder of this paper is organized as follows. Section 2 contains some facts on the numbers \( T'(n, G) \), where \( G \) is a cycle. We first establish the exact value of \( T'(n, C_k) \), where \( k \leq n \leq 2k - 2 \). Next, we continue in this fashion to obtain an upper bound for \( T'(2k-1, C_k) \). Section 3 contains our main result that \( R(P_3, C_k, C_k) = R(C_k, C_k) = 2k-1 \), where \( C_k \) is the odd cycle on \( k \) vertices. The last Section 4 presents two new formulas for the following Ramsey numbers: \( R(P_4, P_4, C_k) = k + 2 \) and \( R(P_3, P_5, C_k) = k + 1 \).

## 2 Values of \( T'(n, C_k) \)

First, we present some facts which are often used in the paper.

**Definition 1** The circumference \( c(G) \) of a graph \( G \) is the length of its longest cycle.

**Definition 2** The girth of a graph \( G \) is the length of its shortest cycle.

**Definition 3** A graph is called weakly pancyclic if it contains cycles of every length between the girth and the circumference.

**Theorem 4 (Brandt, [3])** A non-bipartite graph \( G \) of order \( n \) and more than \( \frac{(n-1)^2}{4} + 1 \) edges contains all cycles of length between 3 and the length of the longest cycle (thus such a graph is weakly pancyclic of girth 3).

**Theorem 5 (Brandt, [4])** Every non-bipartite graph \( G \) of order \( n \) with minimum degree \( \delta(G) \geq (n+2)/3 \) is weakly pancyclic of girth 3 or 4.

The following notation and terminology comes from [6].

For positive integers \( a \) and \( b \) define \( r(a, b) \) as

\[
r(a, b) = a - b \left\lfloor \frac{a}{b} \right\rfloor = a \mod b.
\]
For integers $n \geq k \geq 3$, define $w(n, k)$ as
\[ w(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1), \]
where $r = r(n-1, k-1)$.

Woodall's theorem [12] can then be written as follows.

**Theorem 6 ([6])** Let $G$ be a graph on $n$ vertices and $m$ edges with $m \geq n$ and $c(G) = k$. Then
\[ m \leq w(n, k) \]
and this result is the best possible.

First, we state the following lemma.

**Lemma 7** If $n \geq 2k - 3$ and $k \geq 1$, then $T^*(kK_2, n) = (k-1)n - (k-1)^2$.

**Proof.** The proof is by induction on $k$. It is clear that $T^*(K_2, n) = 0$ for any integer $n$. It is easy to see that $K_{1,r}$ for $r \geq 1$ and $K_3$ are the only connected graphs which do not contain $K_2 \cup K_2$. Thus we obtain $T^*(2K_2, n) = n - 1$ for all $n$, since $K_3$ is not bipartite.

Let $G$ be any nonempty bipartite graph of order $n$, which does not contain $kK_2$. Choose any edge $vw$. Define $H$ to be the subgraph induced by $V(G) - \{v, w\}$. Obviously $H$ cannot contain $(k-1)K_2$, so by the induction hypothesis $e(H) \leq (k-2)(n-2) - (k-2)^2$. Since $G$ is bipartite, so the number of edges with at least one vertex in $\{v, w\}$ is not greater than $n - 1$. Thus we obtain $e(G) \leq (k-2)(n-2) - (k-2)^2 + (n-1) = (k-1)n - (k-1)^2$. Since $K_{k-1,n-k+1}$ implies that $T^*(kK_2, n) \geq (k-1)n - (k-1)^2 = (k-1)(n-k+1)$.

**Lemma 8** Let $G$ be a bipartite graph of order $2k - 2$ with $k^2 - 3k + 4$ edges, where $k$ is odd and $k \geq 9$. Then any two vertices, which belong to different sides of the bipartition, are joined by a path of length $k - 2$.

**Proof.** Let $\{X, Y\}$ be the bipartition of $G$ and choose any two vertices $x \in X, y \in Y$. Graph $G$ can be seen as a complete bipartite graph without at most $k - 3$ edges. Define $X' = (X \setminus \{x\}) \cap N(y)$ and $Y' = (Y \setminus \{y\}) \cap N(x)$. The number of edges in $G$ guarantees that $|X'| \geq 1$, $|Y'| \geq 1$ and $|X'| + |Y'| \geq 2k - 4 - (k - 3) = k - 1$. Thus the complete bipartite graph with bipartition $\{X', Y'\}$ contains at least $k - 2$ edges, so at least one of them, say $vw$, where $v \in X'$ and $w \in Y'$ must belong to $G$ as well. In this way we obtain path $xvwvy$, which is a path of length 3 joining $x$ and $y$. Now we will show that any path of length at least 3 but shorter than $k - 2$ which joins $x$ and $y$ can be extended by two additional vertices to a longer path joining $x$ and $y$, which by induction completes the proof.

Assume that $x$ and $y$ are joined by a path $P$ of length $k - p$, where $4 \leq p \leq k - 3$. Define $G' = G[V(G) \setminus V(P)]$. We have $e(G') = e(G) - e(P) - |\{vw \in E(G) : v \in P, w \in$
$G') \geq k^2 - 3k + 4 - (k - p + 1)^2/4 - (k - p + 1)(k + p - 3)/2$. From Lemma 7 we have $T^*((p/2 + 1)K_2, k + p - 3) = (p^2 + 2kp - 6p)/4$. One can easily verify that this implies $e(G') \geq T^*((p/2 + 1)K_2, k + p - 3)$ and thus $G'$ contains $p/2 + 1$ independent edges. Assume that there is no path of order $k - p + 2$ joining $x$ and $y$ in graph $G$. In this case any edge from $G'$ can be connected to at most $(k - p + 1)/2$ vertices from $P$ or in other words cannot be connected to at least $(k - p + 1)/2$ vertices from $P$. So we have $e(G) \leq e(K_{k-1,k-1}) - \{vw \notin E(G) : v \in P, w \in G'\} \leq (k - 1)^2 - (p/2 + 1)(k - p + 1)/2 = k^2 - (10 + p)k/4 - (p^2 + p + 2)/4 < k^2 - 3k + 4 = e(G)$, a contradiction. Hence there must be a path of order $k - p + 2$ joining $x$ and $y$ in graph $G$. \hfill \Box

Theorem 9 For odd integers $k \geq 5$

$$T'(n, C_k) = w(n, k - 1),$$

where $k \leq n \leq 2k - 2$.

Proof. The last part of the thesis of Theorem 6 implies that $T'(n, C_k) \geq w(n, k - 1)$. Let us suppose that there exists a non-bipartite $C_k$-free graph $G'$ on $n$ vertices which has more than $w(n, k - 1)$ edges. Observe that $w(n, k)$ is not a decreasing function of $k$ and of $n$, i.e. $w(n, k_1) \geq w(n, k_2)$ if $k_1 > k_2$ and $w(n_1, k) \geq w(n_2, k)$ if $n_1 > n_2$. Then, graph $G'$ must contain a cycle of length greater than $k$. Now, we prove that $w(n, k - 1) + 1 > \frac{(n-1)^2}{4} + 1$. The maximal possible value of $n$ is $2k - 2$. Then, the left-hand side is equal to $k^2 - 3k + 4$ and the right-hand side is equal to $k^2 - 3k + \frac{13}{4}$, so by Brandt’s theorem graph $G'$ contains $C_k$. For the case $n = 2k - 3$ we obtain that $r(n - 1, k - 2) = 0$ and $w(n, k - 1) + 1 > \frac{(n-1)^2}{4} + 1$, and $G'$ also contains a cycle of length $k$. For the case $n \leq 2k - 4$ we have that $r(n - 1, k - 2) = n - (k - 1)$. Then, $w(n, k - 1) + 1 = \frac{1}{2}n^2 + k^2 - kn - 3k + \frac{3}{2}n + 3$ and the inequality $w(n, k - 1) + 1 > \frac{(n-1)^2}{4} + 1$ implies the following inequality: $\frac{n^2}{4} + n(2 - k) + k^2 + \frac{7}{4} > 3k$. The minimal value of the left-hand side holds for $n = 2k - 4$ and it is equal to $4k - 2.25$, so for $k \geq 3$ graph $G'$ contains a cycle of length $k$. \hfill \Box

Theorem 10 For odd integers $k \geq 9$

$$T'(2k - 1, C_k) \leq \frac{(2k - 2)^2}{4} - 1 = (k - 1)^2 - 1.$$

Proof. Let $G$ be a non-bipartite graph of order $2k - 1$. By Theorem 4 and by property $w(2k - 1, k - 1) = k^2 - 3k + 5 < \frac{(2k-2)^2}{4} + 2$ we obtain that if $G$ has at least $\frac{(2k-2)^2}{4} + 2$ edges, then it contains $C_k$.

Assume that $G$ has $\frac{(2k-2)^2}{4} + 1 = k^2 - 2k + 2$ edges. Suppose that there is a vertex $v \in V(G)$ such that $d(v) \leq k - 2$. If $G - v$ is a non-bipartite subgraph, we immediately
obtain a contradiction with $T'(2k - 2, C_k) = k^2 - 3k + 3$, so $G - v$ must be bipartite. It is clear that vertex $v$ must be joined to at least one vertex in each side of the bipartition of $G - v$. Applying Lemma 8 we find a cycle $C_k$ in graph $G$, so we have that $\delta(G) = k - 1$. In this case, the number of edges of graph $G$ is at least $rac{(2k-1)(k-1)}{2} = k^2 - \frac{3}{2}k + \frac{1}{2} > k^2 - 2k + 2$, a contradiction. These observations lead us to the conclusion that a non-bipartite graph $G$ on $2k - 1$ vertices and $(\frac{2k-2}{4}) + 1$ edges must contain a cycle $C_k$.

Consider the last case when $G$ has $(k - 1)^2$ edges. Since $w(2k - 1, k - 1) < (k - 1)^2$ for $k > 4$ and $w(k, n)$ is a non-decreasing function of $k$ and $n$, graph $G$ must contain a cycle of length at least $k$. It follows that $\delta(G) \geq k - 2$. We obtain this property using the same arguments as those in the previous case. Since $k - 2 \geq (2k + 1)/3$ for $k \geq 7$, then by Theorem 5 graph $G$ is weakly pancyclic of girth 3 or 4, so it contains a cycle of length $k$.

Finally, for the sake of completeness we recall a few Turán numbers for short paths. In 1975 Faudree and Schelp proved

**Theorem 11 ([9])** If $G$ is a graph with $|V(G)| = kt + r$, $0 \leq r < k$, containing no path on $k + 1$ vertices, then $|E(G)| \leq t\left(\binom{k}{2}\right) + \binom{r}{2}$ with equality if and only if $G$ is either $(tK_k) \cup K_r$ or $((t - l - 1)K_k) \cup (K_{(k-1)/2} + K_{(k+1)/2+l-k+r})$ for some $l$, $0 \leq l < t$, when $k$ is odd, $t > 0$, and $r = (k \pm 1)/2$.

It is easy to check that we obtain the following

**Corollary 12** For all integers $n \geq 3$

$$T(n, P_3) = \left\lfloor \frac{n}{2} \right\rfloor$$

$$T(n, P_4) = \begin{cases} n & \text{if } n \equiv 0 \mod 3 \\ n-1 & \text{otherwise.} \end{cases}$$

$$T(n, P_5) = \begin{cases} \frac{3n}{2} & \text{if } n \equiv 0 \mod 4 \\ \frac{3n}{2} - 2 & \text{if } n \equiv 2 \mod 4 \\ \frac{3n}{2} - \frac{3}{2} & \text{otherwise} \end{cases}$$

### 3 Ramsey numbers for odd cycles

In 1973 Bondy and Erdős proved that

**Theorem 13 ([2])** For odd integers $k \geq 5$

$$R(C_k, C_k) = 2k - 1$$
In 1983 Burr and Erdős gave the following Ramsey number.

**Theorem 14 ([5])**

\[ R(P_3, C_3, C_3) = 11 \]

In 2005 the first author determined two further numbers of this type.

**Theorem 15 ([8])**

\[ R(P_3, C_5, C_5) = 9 \]
\[ R(P_3, C_7, C_7) = 13 \]

Now, we prove our the main result of the paper.

**Theorem 16** For odd integers \( k \geq 9 \)

\[ R(P_3, C_k, C_k) = R(C_k, C_k) = 2k - 1 \]

**Proof.** Let the complete graph \( G \) on \( 2k - 2 \) vertices be colored with two colors, say blue and green, as follows: the vertex set \( V(G) \) of \( G \) is the disjoint union of subsets \( G_1 \) and \( G_2 \), each of order \( k - 1 \) and completely colored blue. All edges between \( G_1 \) and \( G_2 \) are colored green. This coloring contains neither monochromatic (blue or green) cycle \( C_k \) nor a monochromatic (red) path of length 2. We conclude that \( R(P_3, C_k, C_k) \geq 2k - 1 \).

Assume that the complete graph \( K_{2k-1} \) is 3-colored with colors red, blue and green. By Corollary 12, in order to avoid a red \( P_3 \), there must be at most \( k - 1 \) red edges. Suppose that \( K_{2k-1} \) contains at most \( k - 1 \) red edges and contains neither a blue nor a green \( C_k \). Since the number of blue and green edges is greater or equal to \( \binom{2k-1}{2} - (k-1) = 2(k-1)^2 \), at least one of the blue or green color classes (say blue) contains at least \( (k-1)^2 \) edges. If the blue color class is bipartite, then one of the partition sets has at least \( k \) vertices. Since \( R(P_3, C_k) = k \) for \( k \geq 5 \) [11], the graph induced by this partition has to contain a red \( P_3 \) or a green \( C_k \), so blue edges enforce a non-bipartite subgraph of order \( 2k - 1 \) with at least \( (k-1)^2 \) edges which by Theorem 10 contains a blue \( C_k \). \( \square \)

4 The Ramsey numbers \( R(P_l, P_m, C_k) \)

This section makes some observations on 3-color Ramsey numbers for two short paths and one cycle of arbitrary length.

In [1] we can find two values of Ramsey numbers: \( R(P_4, P_4, C_3) = 9 \) and \( R(P_4, P_4, C_4) = 7 \). By using simple combinatorial properties (without the aid of computer calculations) we proved: \( R(P_4, P_4, C_5) = 9 \) and \( R(P_4, P_4, C_6) = 8 \) (see [7] for details).

**Theorem 17**

\[ R(P_4, P_4, C_7) = 9. \]
Proof. The proof of \( R(P_4, P_4, C_7) \geq 9 \) is very simple, so it is left to the reader. Let the vertices of \( K_9 \) be labeled 1, 2, \ldots, 9. Since \( R(P_4, P_4, C_6) = 8 \), we can assume 1, 2, 3, 4, 5, 6 to be the vertices of green \( C_6 \). If the subgraph induced by green edges of \( K_9 \) is bipartite, then since \( R(P_4, P_4) = 5 \), we immediately obtain a red or a blue \( P_4 \). Since \( T(9, P_4) = 9 \), the number of green edges is at least \( 18 > \frac{(9-1)^2}{4} + 1 \), so the non-bipartite subgraph induced by green edges of \( K_9 \) is weakly pancyclic. Since \( R(P_4, P_4, C_3) = 9 \), this subgraph contains green cycles of every length between 3 and the green circumference. Avoiding a green cycle \( C_7 \) we know that the number of green edges from vertices 7, 8, 9 to the green cycle is at most 3. We have to consider the two following cases.

1. There is a vertex \( v \in \{7, 8, 9\} \) which has three green edges to the vertices of green cycle \( C_6 \). We can assume that the edges \( \{1, 7\}, \{3, 7\}, \{5, 7\} \) are green. In this case the edges \( \{2, 4\}, \{4, 6\}, \{2, 6\} \) are red or blue. Without loss of generality we can assume that \( \{2, 4\} \) and \( \{4, 6\} \) are red. This enforces \( \{2, 7\}, \{6, 7\} \) to be blue and \( \{2, 8\}, \{6, 8\} \) to be green, and we obtain a green cycle of length 8 and then a green \( C_7 \).

2. There is a vertex \( v \in \{7, 8, 9\} \) which has two green edges to the vertices of green cycle \( C_6 \). We have to consider two subcases.

   (i) The edges \( \{1, 7\}, \{3, 7\} \) are green and \( \{2, 7\}, \{4, 7\}, \{5, 7\}, \{6, 7\} \) are red or blue. This enforces \( \{2, 6\} \) and \( \{2, 4\} \) to be red or blue. We obtain two situations. In the first, if edge \( \{2, 6\} \) is red and \( \{2, 4\} \) blue, then we can assume that edge \( \{2, 7\} \) is blue, then \( \{5, 7\} \) is red and we obtain a red or a blue \( P_4 \) with edge \( \{6, 7\} \). In the second, if edges \( \{2, 6\} \) and \( \{2, 4\} \) are red, then \( \{4, 7\}, \{6, 7\} \) are blue and \( \{4, 8\}, \{6, 8\}, \{4, 9\}, \{6, 9\} \) are green. Edge \( \{2, 6\} \) cannot be green. If edge \( \{5, 8\} \) is red, then we obtain a blue \( P_4: 2 - 5 - 7 - 6 \) and if \( \{5, 8\} \) is blue, then we have a red \( P_4: 6 - 2 - 5 - 7 \).

   (ii) The edges \( \{1, 7\}, \{4, 7\} \) are green and \( \{2, 7\}, \{3, 7\}, \{5, 7\}, \{6, 7\} \) are red or blue. Then vertex 8 and vertex 9 have green edges to at most one vertex from \( \{2, 3, 5, 6\} \), otherwise we have either the situation considered in (i) or a green cycle of length 8. By simple considering red and blue edges from \( \{7, 8, 9\} \) to \( \{2, 3, 5, 6\} \), we obtain a red or a blue \( P_4 \).

We obtain that there are at least 15 non-green edges from \( \{7, 8, 9\} \) to the vertices of the green \( C_6 \). We can assume that there are at least 8 blue edges among them and we immediately have a blue \( P_4 \). \qed

Theorem 18 For all integers \( k \geq 6 \)

\[ R(P_4, P_4, C_k) = k + 2. \]
Proof. The critical coloring which gives us the lower bound \( k + 2 \) is easy to obtain, so we only give a proof for the upper bound. This proof can be easily deduced from Turán numbers and the theorems given above. By Theorem 9 and Corollary 12 we obtain that \( T'(k+2, C_k) = \frac{1}{2}k^2 - \frac{3}{2}k + 7 \) for \( k \geq 5 \) and \( T(k+2, P_4) \leq k + 2 \). It is easy to check that \( T'(k+2, C_k) \) is greater than the maximal number of edges in a bipartite graph on \( k + 2 \) vertices, thus \( T(k+2, C_k) = T'(k+2, C_k) \). Suppose that we have a 3-coloring of the complete graph \( K_{k+2} \). This graph has \( \frac{1}{2}k^2 + \frac{3}{2}k + 1 \) edges. Note that \( T(k+2, C_k) + 2T(k+2, P_4) \leq \frac{1}{2}k^2 + \frac{1}{2}k + 11 < \frac{1}{2}k^2 + \frac{3}{2}k + 1 \) for all \( k > 10 \). If \( k \in \{8, 9, 10\} \), we obtain that \( T(k+2, C_k) + 2T(k+2, P_4) \leq \binom{k+2}{2} \) with equality for \( k = 8 \) and \( k = 10 \), respectively. So \( R(P_4, P_4, C_9) = 11 \). By Theorem 11 we know the properties of the extremal graphs with respect to \( P_4 \) and by Theorem 9 and [6] we can describe the extremal graphs with respect to \( C_k \), so it is easy to check that the theorem holds for the remaining cases when \( k \in \{8, 10\} \). \( \square \)

The following lemma will be useful in further considerations.

Lemma 19 Suppose that graph \( G \) has \( k+1 \) vertices and it contains a cycle \( C_k \) and suppose that we have a vertex \( v \notin V(C_k) \), which is joined by \( r \) edges to \( C_k \), where \( 2 \leq r \leq k \). Then one of the following two possibilities holds:

(i) \( G \) contains a cycle \( C_{k+1} \).

(ii) \( G' = G[V(C_k)] \) contains at most \( \frac{k(k-1)}{2} - \frac{r(r-1)}{2} \) edges.

Proof. Let \( C = x_1x_2x_3...x_k \) be a cycle \( C_k \) and \( v \notin V(C) \) be a vertex, which is joined by \( d(v) = r \) edges to \( C \), where \( 2 \leq r \leq k \). First, if \( r \geq \lceil \frac{k}{2} \rceil \), then we immediately have a cycle \( C_{k+1} \) and (i) follows. Assume that \( 2 \leq r \leq \lceil \frac{k}{2} \rceil - 1 \). Let the vertices of \( C \), which are joined by an edge to vertex \( v \), be labeled \( p_{i_1}, p_{i_2}, ..., p_{i_r} \). If any two of them are adjacent, then we obtain the cycle \( C_{k+1} \) and (i) follows. Otherwise, consider the following vertices: \( p_{i_1+1}, p_{i_2+1}, ..., p_{i_r+1} \). In order to avoid a cycle \( C_{k+1} \), these vertices must be mutually nonadjacent and \( G' \) contains at most \( \frac{k(k-1)}{2} - \frac{r(r-1)}{2} \) edges. \( \square \)

Theorem 20 For all integers \( k \geq 8 \)

\[ R(P_3, P_5, C_k) = k + 1. \]

Proof. A critical coloring which gives us the lower bound \( k + 1 \) is very simple, so all we need is the upper bound. It is easy to see that simply using Turán numbers does not give us the proof. Indeed, the sum \( T(k+1, P_3) + T(k+1, P_5) + T(k+1, C_n) \) is far greater than the maximal number of edges in the complete graph on \( k + 1 \) vertices. Suppose that we have a 3-coloring of \( K_{k+1} \) with colors red, blue and green which neither contains a red \( P_3 \), nor a blue \( P_5 \), nor a green \( C_k \). \( K_{k+1} \) has to contain a green cycle \( C_{k-1} \). Indeed, suppose...
that this is not the case. Since \( T(k+1, P_3) + T(k+1, P_5) + T(k+1, C_{k-1}) < \binom{k+1}{2} \) for \( k > 11 \), we obtain either a red \( P_3 \) or a blue \( P_5 \). For the case of \( k \in \{8,9,10,11\} \) we use the properties of the extremal graphs with respect to \( P_3 \) and \( P_5 \) and we also obtain either a red \( P_3 \) or a blue \( P_5 \). Let the vertices of \( K_{k+1} \) be labeled \( v_0, v_1, \ldots, v_k \). We can assume the first \( k-1 \) vertices to be the vertices of green \( C_{k-1} \). It is easy to see that \( b(v_{k-1}) \) and \( b(v_k) \) are greater or equal to \( k - \lfloor (k-1)/2 \rfloor - 1 \). Note that in order to avoid a blue \( P_5 \) we obtain that the vertices \( v_{k-1} \) and \( v_k \) have no common vertex which belongs to \( V(C_{k-1}) \) and which is joined by a blue edge to them. If the vertex \( v_{k-1} \) or \( v_k \) is joined by at least 4 green edges to the vertices of \( C_{k-1} \), then by Lemma 19 and \( R(P_3, P_5) = 5 \) we have a blue \( P_5 \). If \( v_{k-1} \) and \( v_k \) are joined by at most 3 green edges to the vertices of \( C_{k-1} \), then by Lemma 19 and \( R(P_3, P_4) = 4 \) we obtain a blue \( P_4 \). If \( k \geq 9 \) then we also have a blue \( P_5 \). In the case \( k = 8 \) by simple considering possible colorings of the edges of \( v_{k-1} \) and \( v_k \) we obtain either a red \( P_3 \), or a blue \( P_5 \), or else a green \( C_k \).

\[ \square \]

References

[1] Arste J., Klamroth K., Mengersen I.: Three color Ramsey numbers for small graphs, 


