On the deficiency of bipartite graphs

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Abstract

Given a graph $G$, an edge-coloring of $G$ with colors $1, 2, 3, \ldots$ is consecutive if the colors of edges incident to each vertex form an interval of integers. This paper is devoted to bipartite graphs which do not have such a coloring of edges. We investigate their consecutive coloring deficiency, or shortly the deficiency $d(G)$ of $G$, i.e. the minimum number of pendant edges whose attachment to $G$ makes it consecutively colorable. In particular, we show that there are bipartite graphs whose deficiency approaches the number of vertices. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we investigate a new graph invariant, namely the consecutive (or interval) coloring deficiency of a graph. Given a coloring of the edges of graph $G$ with colors $1, 2, 3, \ldots$ the coloring is said to be a consecutive coloring if the colors received by the edges incident to each vertex are distinct and form an interval of integers. Not all graphs have such a coloring. A simple counterexample is $K_3$. However, every graph $G$ which is not consecutively colorable can be augmented to a supergraph of $G$ which has such a coloring by the attachment of some pendant edges to its vertices. The minimum number of edges whose attachment to $G$ renders it consecutively colorable is just the deficiency of $G$, in short $d(G)$.

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The consecutive coloring of the edges has immediate applications in scheduling theory, in the case when a minimum length schedule without waiting periods and idle times is searched for. In this application the vertices of a graph correspond to processors or jobs, edges represent unit execution time tasks and colors correspond to assigned time units. Consider, for example, The Southeastern Public Interest Job Fair which is held each year during a weekend in November, when 25–50 law firms from all over the US and 100–200 law students from the Southeast come to Atlanta for job interviews [2]. To arrange all meetings between the firms and students we can model the scheduling problem as a bipartite graph coloring problem. However, since the students have fewer meetings than law firms, it is possible for an edge coloring algorithm to solve the problem in such a way that many students complain their meetings have not been scheduled consecutively. So the following questions arise naturally. Is it possible to create a no-wait schedule for both students and firms? If not, what is the deficiency of the underlying scheduling graph \( G \)? What is the minimum number of vertices of \( G \) which have nonconsecutive colors?

The consecutive edge-coloring problem apparently was first studied under the name of “interval coloring” by Asratian and Kamalian [1,10], Hanson and Loten [6,7] and Sevastjanov [11]. Their work was devoted to bipartite graphs only. Most of these papers, however, dealt with sufficient conditions under which bipartite graphs have a consecutive coloring of edges [5–7,10]. The others showed the NP-completeness of deciding whether a given bipartite graph admits a consecutive coloring [3,11]. In contrast to these our paper investigates bipartite graphs not allowing a consecutive coloring of edges.

If a bipartite graph \( G=(V_1, V_2; E) \) is not consecutively colorable, then we can ask a question about consecutive colorability with a fixed number of colors for the vertices in \( V_1 \). Because of the above-mentioned analogy with the open shop scheduling colorability problem we shall call the vertices of \( V_1 \) as processors. A \( V_1 \)-sided consecutive coloring is defined as a coloring in which colors received by the edges incident to each processor form an interval of integers. Asratian and Kamalian [1] proved that given a bipartite graph \( G=(V_1, V_2; E) \), there is a \( V_1 \)-sided consecutive \( t \)-coloring of \( G \) for every \( k \leq t \leq |E| \), where \( k \) is the least value of \( t \) for which a \( V_1 \)-sided consecutive coloring of \( G \) exists. However, finding the value of \( k \) is NP-hard (cf. [9]).

The deficiency of graph \( G \) seems first studied by the authors in [4]. The cited paper established the deficiency of odd cycles, wheels and almost wheels, and complete graphs of odd order. In the present paper we investigate the deficiency of bipartite graphs. In Section 2 we give basic definitions and facts concerning consecutive edge colorings and deficiency of graphs in general and in the case of bipartite graphs, in particular. In Section 3 we show that bipartite graphs with at most 3 processors all have consecutive colorings of edges. Section 4 is devoted to the deficiency of 4-processor bipartite graphs of special regular structure which we call rosettes. We give there the smallest currently known bipartite graph with positive deficiency. In Section 5 we study the deficiency of bipartite graphs defined by Hertz [8] and show that their deficiency approaches the number of vertices.
2. Basic definitions and results

All graphs considered in this paper are simple and connected. Given such a graph \(G = (V(G), E(G))\), by \(n(G)\) and \(\Delta(G)\) we denote the number of vertices and maximum vertex degree in \(G\), respectively. We shall drop the reference to the graph, writing \(n\) and \(\Delta\), if \(G\) is clear from the context. By \(\mathbb{N}\) we denote the set of positive integers.

**Definition 2.1.** Given a finite subset \(A\) of \(\mathbb{N}\), by the deficiency \(d(A)\) of \(A\) we mean the cardinality of the smallest set \(B \subset \mathbb{N}\) such that \(A \cup B\) is an interval.

**Definition 2.2.** Given a graph \(G = (V(G), E(G))\) and its edge-coloring \(c : E(G) \rightarrow \mathbb{N}\), by the deficiency of \(c\) at vertex \(v \in V(G)\), \(d(G,c,v)\), we mean the deficiency of the set of colors of edges incident with \(v\). The deficiency of coloring \(c\) is the sum of deficiencies of all vertices in \(G\), in symbols \(d(G,c) = \sum_{v \in V(G)} d(G,c,v)\). By the deficiency of graph \(G\), \(d(G)\), we mean the minimum \(d(G,c)\) among all possible colorings \(c\) of \(G\), and we say that \(G\) is \(d(G)\)-deficient.

According to Definition 2.2, graphs \(G\) having a consecutive coloring are \(0\)-deficient. For this reason we shall call them graphs of Class 0. It is easy to see that \(d(G)\) is equal to the minimum number of pendant edges whose attachment to the vertices of \(G\) makes a Class 0 graph. Note that given \(k \geq 0\) and a graph \(G\), the problem of deciding if \(d(G) \leq k\) is \(NP\)-complete since it is already \(NP\)-complete to decide if \(G\) is \(0\)-deficient. In fact, Sevastjanov [11] proved that deciding if \(G\) is Class 0 is strongly \(NP\)-complete even when \(G\) is bipartite.

If \(G\) is simple then by Vizing’s theorem [12] \(\chi'(G) \leq \Delta + 1\), so because \(\Delta \leq \chi'(G)\), we have only two possibilities for the chromatic index of \(G\). Accordingly, \(G\) is said to be Class 1 if \(\chi'(G) = \Delta\), and \(G\) is said to be Class 2 if \(\chi'(G) = \Delta + 1\). Well-known examples of Class 1 graphs are bipartite graphs. Asratian and Kamalian [1] showed that all graphs of Class 0 belong to Class 1. However, the converse is not true even for bipartite graphs, since there are graphs that are not consecutively colorable. The smallest known example, in the sense of \(d(G)\), is shown in Fig. 1.

**Definition 2.3.** Let \(G\) be a Class 0 graph. By the span \(s(G,c)\) of coloring \(c\) we mean the number of colors used in the consecutive coloring. By \(\chi'_c(G)\) and \(S(G)\) we denote the minimum and maximum span among all consecutive colorings of \(G\), respectively. The minimum span \(\chi'_c(G)\) will be called the consecutive chromatic index of \(G\).

Clearly, \(\Delta(G) \leq \chi'_c(G) \leq S(G)\). In [1] it was shown that if \(G\) is a Class 0 bipartite graph, then \(S(G) \leq n - 1\), and this bound is tight for paths \(P_n\) and complete bipartite graphs \(K_{r,s}\).

Examples of Class 0 bipartite graphs are: trees, \(\Delta\)-regular and grid graphs for which \(\chi'_c(G) = \Delta\) [3,10], complete graphs \(K_{r,s}\) with \(\chi'_c(K_{r,s}) = r + s - \gcd(r,s)\) [6,10] as well as subcubic, cacti and \((2,\Delta)\)-regular graphs for which \(\chi'_c(G) = \Delta\) or \(\Delta + 1\) [5,7].
Kamalian [10] showed the following property of trees and complete bipartite graphs: for any \( t \in \{ \chi''(G), \ldots, S(G) \} \) there is a consecutive coloring \( c \) such that \( s(G, c) = t \). This property does not hold for general bipartite graphs. Sevastjanov [11] gave an example of a bipartite graph whose span takes on the values of 100 and 173 only.

In contrast to \( \chi'(G) \) the consecutive chromatic index is not bounded in terms of \( \Delta(G) \). In fact, for each \( k \in \mathbb{N} \) there exists a bipartite graph \( G_k \) such that \( \Delta(G_k) = 32 \) and \( \chi'_c(G_k) \geq 2k - 1 \), so \( \lim_{k \to \infty} (\chi'_c(G_k)/\Delta(G_k)) = \infty \) (cf. [4]). Thus \( \chi'_c(G) \) is not directly related to \( \chi'(G) \) even if \( G \) is bipartite.

### 3. 2- and 3-processor graphs

Let \( G = (V_1, V_2; E) \) be a 2-processor graph, i.e. such that \( |V_1| = 2 \). Because of its characteristic structure it is sometimes called a candy (cf. [13]). An example of a candy is shown in Fig. 2.

It is easy to see that the core of \( G \), i.e. the subgraph obtained by deleting all pendant edges is a complete bipartite graph \( K_{2,s} \). Since the core of \( G \) is consecutively colorable so is a 2-processor graph \( G \).

Now suppose that \( G \) is a 3-processor graph. We have the following.
Theorem 3.1. Any 3-processor graph $G$ belongs to Class 0.

Proof. Without loss of generality we assume that $G = (V_1, V_2; E)$ has no pendant vertices. Let the processors of $V_1$ be denoted by $p_1, p_2, p_3$ in such a way that the subset of vertices of degree 2 adjacent to $p_1$ and $p_3$ is as small as possible. Let $a$ be the cardinality of this set. From $V_2$ we choose any 3 vertices of degree 2 that are adjacent to $p_1$ and $p_3$, $p_2$ and $p_3$, $p_1$ and $p_2$, respectively. The edges incident to the chosen vertices constitute a cycle $C_6$ in which every other vertex is $p_1$, $p_2$ and $p_3$. We color the edges of $C_6$ consecutively, as shown in Fig. 3. After that we choose another 3 vertices of degree 2 from $V_2$ which are adjacent to the corresponding processors and color their edges with colors of value two more than previously so that the colors at $p_1$, $p_2$, $p_3$ are $\{3, 4\}$, $\{4, 5\}$ and $\{5, 6\}$, respectively. We repeat this until there are no more uncolored edges on vertices of degree 2 adjacent to $p_1$ and $p_3$. The sets of colors used on edges incident to $p_1, p_2, p_3$ form intervals $\{1, \ldots , 2a\}$, $\{2, \ldots , 2a + 1\}$, and $\{3, \ldots , 2a + 2\}$.

Next we color the edges of subgraph induced by $V_1$ and all vertices of degree 3 in $V_2$. Let $b$ be the number of such vertices in $V_2$. It is easy to see that the subgraph is $K_{3,b}$ and it can be colored consecutively with colors $\{1, \ldots , b\}$, $\{2, \ldots , b + 1\}$, $\{3, \ldots , b + 2\}$ at vertices $p_1$, $p_2$, $p_3$, respectively. Increasing the color of each edge by $2a$ we obtain a consistent consecutive coloring of a larger subgraph of $G$ using colors $\{1, \ldots , 2a + b\}$, $\{2, \ldots , 2a + b + 1\}$, $\{3, \ldots , 3a + b + 2\}$. Finally, we extend the coloring on these yet uncolored edges at vertices of degree 2 in $V_2$ which are incident to $p_1, p_2$ or $p_2, p_3$. The first subset of edges forms a complete bipartite graph which can be colored consecutively so that the smallest color at $p_1$ is $2a - b + 1$ and the smallest color at $p_2$ is $2a + b + 2$. Similarly, the second set of edges forms a complete bipartite graph which can be colored consecutively (possibly using negative colors) so that the largest color at $p_2$ is 1 and the largest color at $p_3$ is 2. Altogether three edge colorings (possibly shifted above by a positive constant) give a consecutive coloring of the whole of $G$. The complexity of this method is linear and the number of colors used is at most $A + 2$. □

Motivated by the result of Theorem 3.1 we implemented a computer program which calculated the deficiency of all 4-processor graphs on at most 19 vertices. The program...
confirmed that the rosette $M_5$ of Fig. 1 is the smallest 4-processor graph with positive deficiency.

4. Two rosettes

The following theorem holds for general graphs but it is a very useful tool in estimating the deficiency of certain bipartite graphs.

**Theorem 4.1.** If there are sequences of positive integers $p_1, p_2, \ldots, p_l$ and $q_1, q_2, \ldots, q_l$ and there is a vertex $v \in V(G)$ of degree $deg(v) > 1$ in $G$ such that for any two vertices $u \neq w$ adjacent to $v$ there is a path $u = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p = w$, where $p_i$ is one of the numbers of the first sequence, such that

$$\deg(v) + p_i - 1 \geq q_i + \sum_{j=1}^{p_i} \deg(v_j),$$

then

(i) $d(G) \geq \min q_i$, $i = 1, \ldots, l$,

(ii) for any coloring $c$ of graph $G$ there is a vertex $v$ such that $d(G, c, v) \geq \min \{q_i/p_i\}$.

**Proof.** Let $c$ be a coloring of $G$ and let $u, w$ be two vertices adjacent to $v$ such that $c(u, v) = c_{\min}(v)$ and $c(w, v) = c_{\max}(v)$, where $c_{\min}(v)$ ($c_{\max}(v)$) is the minimum (maximum) color number of an edge on $v$. By assumption, there is a path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p$, connecting $u = v_1$ and $w = v_p$, such that

$$\deg(v) + p_i - 1 \geq q_i + \sum_{j=1}^{p_i} \deg(v_j).$$

To simplify the notation we assume that $v = v_0 = v_{p+1}$ and $d(x) = d(G, c, x)$. Then

$$c_{\max}(v) - c_{\min}(v) = \sum_{j=1}^{p_i} (c(v_{j+1}, v_j) - c(v_j, v_{j-1})) \leq \sum_{j=1}^{p_i} (c_{\max}(v_j) - c_{\min}(v_j)).$$

By the definition of deficiency, for any $x \in V(G)$

$$d(x) = c_{\max}(x) - c_{\min}(x) - \deg(x) + 1,$$

hence

$$\deg(v) - 1 \leq c_{\max}(v) - c_{\min}(v) \leq \sum_{j=1}^{p_i} (c_{\max}(v_j) - c_{\min}(v_j))$$

$$= \sum_{j=1}^{p_i} (\deg(v_j) - 1) + \sum_{j=1}^{p_i} d(v_j) = \sum_{j=1}^{p_i} \deg(v_j) + \sum_{j=1}^{p_i} d(v_j) - p_i.$$
and
\[ q_i \leq \sum_{j=1}^{p_i} d(v_j). \]

But
\[ d(G,c) \geq \sum_{j=1}^{p_i} d(v_j) \geq q_i, \]

which proves (i).

Moreover, one of the values of \( d(v_j), j = 1, \ldots, p_i \) must be at least \([q_i/p_i]\), which implies (ii). □

In particular, for \( l = 1 \) we have

**Corollary 4.2.** If there are \( p, q \in \mathbb{N} \) and vertex \( v \in V(G) \) with \( \deg(v) > 1 \) and for any two vertices \( u \neq w \) adjacent to \( v \) there is a path \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_p \) connecting \( u \) and \( w \) (i.e. \( v_1 = u \) and \( v_p = w \)) such that
\[ \deg(v) + p - 1 \geq q + \sum_{j=1}^{p} \deg(v_j), \]

then

(i) \( G \) has the deficiency \( d(G) \geq q \).

(ii) for any \( c \) there is a vertex \( v \in V(G) \) with \( d(G,c,v) \geq \lceil q/p \rceil \). □

Now we consider a sequence of 4-processor bipartite graphs which we call Małafiejski's rossettes. The \( k \)th rossette \( M_k \) is shown in Fig. 4 (see also Fig. 1). The following theorem estimates \( d(M_k) \) for each \( k \) and shows that the deficiency at a vertex in \( V(M_k) \) can be arbitrarily high.

**Theorem 4.3.** For each \( k \geq 2 \)
\[ k - 4 \leq d(M_k) \leq k - 2. \quad (4.1) \]

Moreover, for each \( l \in \mathbb{N} \) and \( k > 3l + 4 \) and for any coloring \( c \) of \( M_k \) there is a vertex \( v \in V(M_k) \) such that \( d(M_k,c,v) > l \).

**Proof.** Suppose \( k \geq 4 \). We apply Corollary 4.2 to the central vertex \( a \) of \( M_k \). It is easy to see that \( \deg(a) = 3k \). Note that any two vertices adjacent to \( a \) are connected by a path of two edges whose central vertex is one of \( e, f, g \) of degree \( 2k \). Moreover, the degree of any vertex adjacent to \( a \) is \( 3 \). So Corollary 4.2 holds with \( p=3 \) and \( q=k-4 \), which gives the left-hand inequality of (4.1) and \( l < \lceil (k-4)/3 \rceil \leq d(M_k,c,v) \). To show the right-hand inequality of (4.1) we define a coloring \( c \) of \( M_k \) with \( d(M_k,c) = k - 2 \). Namely,
\[ c(a,b_i) = i, \ c(a,c_i) = k + i, \ c(a,d_i) = 2k + i, \ c(e,b_i) = i + 1, \ c(e,c_i) = k + 2 + i, \ c(f,c_i) = k + 1 + i, \ c(f,b_i) = 2k + 1 + i, \ c(g,b_i) = i + 2, \ c(g,d_i) = 2k - 1 + i \]
for \( i = 1, \ldots, k \). The proof is complete. □
In [11] Sevastjanov presented a bipartite graph with \( n = 28 \), \( \Delta = 21 \) which is 1-deficient. Generalizing his construction we get a sequence of Sevastjanov’s rosettes \( S_1, S_2, \ldots \) with the deficiency tending to infinity. The \( k \)th rosette \( S_k \) is shown in Fig. 5. The original Sevastjanov graph is \( S_7 \) in this sequence.

**Theorem 4.4.** For each \( k \geq 5 \)

\[
k - 6 \leq d(S_k) \leq k - 5.
\]  

**Proof.** Let us consider rosette \( S_k \), \( k \geq 5 \). The left-hand inequality of (4.2) follows from Theorem 4.1 for the central vertex \( a \) of degree \( \deg(a) = 3k \). Note that any vertex
adjacent to \( a \) has degree 2 and any two such vertices \( u \neq w \) can be connected by either:

1. a path of 2 edges whose central vertex is one of \( b, c, d \), each of degree \( k + 2 \). In this case vertices \( u \) and \( w \) belong to the same group, accordingly; or

2. a path of 4 edges going around the external face with succeeding vertices having degree: \( k + 2, k + 2 \). In this case the vertices \( u \) and \( w \) belong to different groups.

Thus the assumptions of Theorem 4.1 are fulfilled for 2-element sequences \((l - 2)\) with \( p_1 = 3\), \( p_2 = 5\) and \( q_1 = 2k - 4\), \( q_2 = k - 6\), respectively. The same theorem gives us the lower bound \( k - 6 \leq d(S_k) \). To complete the proof all we need is showing a \((k - 5)\)-deficient coloring of \( S_k \). This is as follows:

(i) \( k \) is odd

\[

c(a,b_i) = i, \quad c(a,c_i) = k + i, \quad c(a,d_i) = 2k + i, \quad c(b,b_i) = i + 1, \quad c(b,d_i) = 2k - 1 + i.
\]
\[
c(b,f) = k + 2, \quad c(b,g) = k + 3, \quad c(d,g) = k + 4, \quad c(d,e) = 2k - 1, \quad c(c,e) = 2k, \quad c(f) = k.
\]
\[
c(c,c_1) = k + 2, \quad c(c,c_2) = k + 1, \quad c(c,c_3) = k + 4, \quad c(c,c_4) = k + 3, \ldots, \quad c(c,c_{k-1}) = 2k - 1.
\]
\[
c(c,c_{k-1}) = 2k - 2, \quad c(c,c_k) = 2k + 1 \text{ for } i = 1, \ldots, k.
\]

Thus we have

\[
d(S_k) \leq d(S_k,c) = d(S_k,c,d) + d(S_k,c,f) = k - 5.
\]

(ii) \( k \) is even

The coloring is the same except that of the edges incident with vertices \( c \) and \( e \). Namely, we have

\[
c(d,e) = 2k - 2, \quad c(c,e) = 2k - 1, \quad c(f) = k, \quad c(c,c_1) = k + 2, \quad c(c,c_2) = k + 1, \quad c(c,c_3) = k + 4.
\]
\[
c(c,c_4) = k + 3, \ldots, \quad c(c,c_{k-1}) = 2k - 2, \quad c(c,c_k) = 2k - 3, \quad c(c,c_{k-1}) = 2k, \quad c(c,c_k) = 2k + 1
\]

for \( i = 1, \ldots, k \). Analogously as above

\[
d(S_k) \leq k - 5.
\]

5. Hertz’s graphs

In this section we consider a family of graphs which have particularly high values for the deficiency.

Hertz [8] has given a bipartite graph \( G \) with \( n(G) = 23 \), \( \Delta(G) = 14 \) and \( d(G) = 1 \). His construction can be generalized to what we now call the Hertz’s graphs. Fig. 6 shows the idea of this construction. According to this the original Hertz graph can be expressed as \( H_{7,2} \). It is easy to see that \( \Delta(H_{k,1}) = kl \) and \( n(H_{k,1}) = kl + k + 2 \).

**Theorem 5.1.** For any \( k \geq 4 \), \( l \geq 3 \)

\[
d(H_{k,1}) = kl - k - 2l + 2.
\]

**Proof.** Consider a graph \( H_{k,l} \) with \( k \geq 4 \), \( l \geq 3 \). We apply Theorem 4.1 to vertex \( v = a \). Let \( u = c_{ij} \) and \( w = c_{rs} \) \((i,r \in \{1, \ldots, k\}, \ j,s \in \{1, \ldots, l\}\) be any two vertices adjacent to \( a \). If \( i = r \) then \( u \) and \( w \) can be connected by a path of 2 edges, namely \( u \rightarrow d_i \rightarrow w \), in which the degrees of succeeding vertices are 2, \( l + 1 \) and 2. If \( i \neq r \) then we have a path of length 4, namely \( u \rightarrow d_i \rightarrow b \rightarrow d_r \rightarrow w \) with degrees
2, \( l + 1 \), \( k \), \( l + 1 \) and 2, respectively. Hence Theorem 4.1 holds with \( l = 2 \), \( \deg(v) = kl \) as well as \( p_1 = 3 \), \( p_2 = 5 \) and \( q_1 = kl - l - 3 \), \( q_2 = kl - k - 2l - 2 \). Thus we have the lower bound \( kl - k - 2l - 2 \leq d(H_{k,l}) \).

It remains to show a coloring \( c \) with \( d(H_{k,l}, c) = kl - k - 2l - 2 \). This coloring is as follows:

For each \( j = 1, \ldots, l \) \( c(a, c_{i,j}) = (i - 1)l + j \), \( c(d_i, c_{i,j}) = (i - 1) + j \), where the "-" sign applies to \( i = 2, k \) and "+" to the remaining values of \( i \). Finally, \( c(b, d_i) = c(d_i, c_{i,1}) + 1 \) for \( i = 1, 2 \) and \( c(b, d_i) = c(d_i, c_{i,1}) - 1 \) for \( i = 3, \ldots, k \).

Thus we have

\[
d(H_{k,l}) \leq d(H_{k,l}, c) = d(H_{k,l}, c, b) = kl - k - 2l - 2.
\]

Now, let us denote \( G_k = H_{k,k} \). Then \( n(G_k) = k^2 + k + 2 \) and \( d(G_k) = k^2 - 3k - 2 \), which leads to the conclusion that \( \lim_{k \to \infty} \frac{d(G_k)}{n(G_k)} = 1 \).

We conclude our considerations with an open problem concerning the deficiency of \( G \). Let \( d(n) = \max\{d(G) : |V(G)| \leq n\} \). We have just shown that \( d(n) = \Omega(n) \). From Vizing's theorem [12] it follows that for any graph \( G \), \( d(G) < n \Delta < n^2 \). Thus \( d(n) = O(n^2) \). But this bound seems far from tight.

**Problem.** What is the order of growth of \( d(n) \)?

**References**