JOURNAL OF APPLIED
COMPUTER SCIENCE
Vol. xx No. xx (xxxx), pp. xx–xx

Some lower bounds on the Shannon capacity

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Abstract. *In the paper we present a measure of a discrete noisy channel, named the Shannon capacity, which is described in the language of graph theory. Unfortunately, the Shannon capacity C_0 is difficult to calculate, so we try to estimate the value of C_0 for specific classes of graphs, i.e. circular graphs.*
Keywords: *Shannon capacity, circular graphs, information theory*

1. Introduction

Within information theory there exists a broad field of zero-error communication modelling of noisy channels. Informally, its aim is, given the description of the channel, to minimize the amount of data that must be sent to ensure reliable transmission.

This project has been partially supported by Narodowe Centrum Nauki under contract DEC-2011/02/A/ST6/00201.

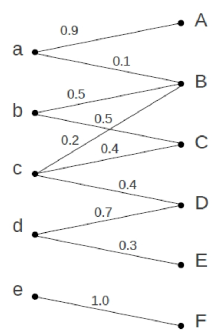


Figure 1: An example of discrete channel with $A_X = \{a, b, c, d, e, f\}$ and $A_Y = \{A, B, C, D, E\}$. If $p(y|x) > 0$, then $x \in A_X$ is adjacent to $y \in A_Y$

Definition 1. *A discrete channel Q is defined as a triple (A_X, A_Y, M_{XY}), where A_X and A_Y are input and output alphabets and M_{XY} is the transition matrix: every element corresponds to a given conditional probability $p(y|x)$.*

An example of discrete channel is shown in Figure 1.

We say that a channel Q is noisy, if there are different elements $y_1, y_2 \in A_Y$ and an element $x \in A_X$ such that $p(y_1|x)p(y_2|x) > 0$. The channel from Figure 1 is noisy. Using the channel definition, we can introduce the following notation:

Definition 2. *We define the channel capacity as*

$$C = \max_p I(X;Y) = \max_p \sum_{x \in A_X} \sum_{y \in A_Y} \frac{p(x,y)}{p(x)p(y)} \tag{1}$$

where X, Y are random variables and maximum is taken over all possible distributions $p(x)$.

In practice, we compute C using the formula above. Operationally, the channel capacity is the highest rate (given in bits per channel) at which information can be sent with arbitrary low probability of error. However, to achieve zero-error probability it can be useful to model the channel in terms of graphs, as it was introduced by Shannon [1].

2. Graph theory and Shannon capacity

We begin this section with definitions of some basic graph-theoretic concepts.

Definition 3. *Let V be a set of vertices. Then $G = (V, E)$ is a directed graph if E is a subset of $V \times V$.*

Throughout this paper graphs are understood to be finite, undirected, and simple (i.e., without self-loops and parallel edges). Moreover, we will denote by $V(G)$ and $E(G)$ the set of vertices and edges of graph G, respectively.

Definition 4. *A graph H is a subgraph of G iff $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$.*

Definition 5. *A graph G is vertex-transitive iff given any two vertices $u, v \in V(G)$, there is some automorphism $f : V(G) \to V(G)$ such that $f(u) = v$.*

Definition 6. *A graph G is edge-transitive iff given any two edges $e_1, e_2 \in E(G)$, there is some automorphism $f : E(G) \to E(G)$ such that $f(e_1) = e_2$.*

Furthermore, in this paper we will use the concept of the strong product of two graphs:

Definition 7. *We define the strong product as follows: given two graphs G and H, the vertices of it are all the pairs in the Cartesian product $V(G) \times V(G)$. There is an edge between (v, v') and (w, w') iff one of the following conditions holds:*

- $\{v, w\} \in E(G)$ *and* $\{v', w'\} \in E(H)$,
- $v = w$ *and* $\{v', w'\} \in E(H)$,
- $v' = w'$ *and* $\{v, w\} \in E(G)$.

See Figure 2 for an example.

We write $G \boxtimes H$ to denote the strong product of G and H, and $G^{\boxtimes n}$ to denote $G \boxtimes G \boxtimes \ldots \boxtimes G$, where G occurs n times.

Let G be a graph. A set of vertices of G is said to be an independent set of vertices if they are pairwise non-adjacent. The independence number of G, denoted by $\alpha(G)$, is defined to be the size of a largest independent set of G. If n is a natural number and G and H are graphs, then for the strong product the following relation holds:

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H) \tag{2}$$

From Equation 2, we know that $\alpha(\cdot)$ is super-multiplicative:

$$\alpha(G^{\boxtimes(n+m)}) \geq \alpha(G^{\boxtimes n})\alpha(G^{\boxtimes m}) \tag{3}$$

where n, m are natural numbers.

Now we may introduce the notion of a measure of a discrete noisy channel:

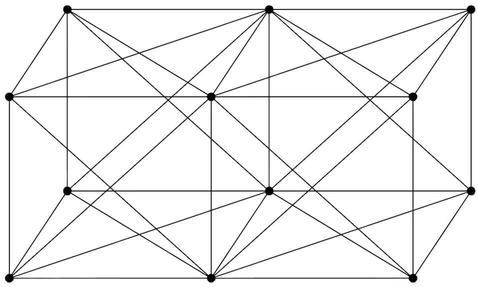


Figure 2: An example of the strong product $P_3 \boxtimes C_4$

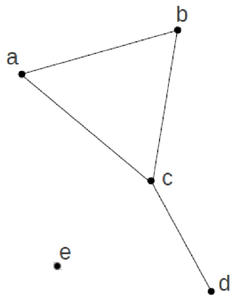


Figure 3: The graph G, associated with the channel presented in Fig. 1

Definition 8. *The effective size of an alphabet in the zero-error transmission is known as the Shannon capacity of a channel. Formally, the Shannon capacity is defined as:*

$$C_0(G) = \sup_{n \geq 1} \sqrt[n]{\alpha(G^{\boxtimes n})} = \lim_{i \to \infty} \sqrt[n]{\alpha(G^{\boxtimes n})} \tag{4}$$

The last equation holds since from Equation 3 $\alpha(\cdot)$ is super-multiplicative.

This definition is derived from a discrete noisy channel Q (with an input alphabet X and an output alphabet Y) modelled using a characteristic graph G: we associate with the vertex set of G the input alphabet X and the edge set of G consists of channel input pairs that may be confused by giving the same output (see Fig. 3).

Calculation of the Shannon capacity is hard: we know that for C_3, C_5 we have the value 1 and $\sqrt{5}$, respectively [2], and for even cycle it is equal to $\alpha(G) = \frac{n}{2}$, but the exact value has not been found even for small cycle C_7.

From the famous theorem by Lovász [2] we know the bounds on the value of Shannon capacity of a given graph G and natural number i:

$$\alpha(G) \le \Theta(G) \le \vartheta(G) \tag{5}$$

for some function (named Lovász number [2]) which is multiplicative

$$\vartheta(G \boxtimes H) = \vartheta(G)\vartheta(H) \tag{6}$$

Theorem 9 (Rosenfeld [3]). *For any two graphs G, H:*

$$\alpha(G \boxtimes H) \le \alpha(G)\alpha^*(H) \tag{7}$$

where $\alpha^*(G)$ is a fractional independence number, a solution of the relaxation of an independence number problem, satisfying $\alpha(G) \le \alpha^*(G)$.

In this paper, we consider graphs with $C_0(G) > \alpha(G)$, because their channels bring a benefit in zero-error transmission [1].

3. Regular graphs

Since the Shannon capacity is bounded from above by Lovász number, the exact value and the upper bounds on Lovász number are useful to approximate the capacity of the channel.

Lovász cite Hoffman giving the upper bound on the value of Lovász number for regular graphs:

Theorem 10 (Lovász [2]). *If G is a regular graph on n vertices and its adjacency matrix has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then*

$$\vartheta(G) \le \frac{-n\lambda_n}{\lambda_1 - \lambda_n}. \tag{8}$$

The equality holds if G is edge-transitive.

By definition, all vertex-transitive graphs must be regular. For these graphs the following upper bound is known.

Theorem 11 (Lovász [2]). *If G is a vertex-transitive graph on n vertices, then*

$$C_0(G)C_0(\bar{G}) \le \vartheta(G)\vartheta(\bar{G}) = n \tag{9}$$

From this Theorem straightly follows the Shannon capacity for self-complementary graphs:

Corollary 12. *If G is a vertex-transitive self-complementary graph on n vertices, then*

$$C_0(G) = \vartheta(G) = \sqrt{n}. \tag{10}$$

Another theorem, which gives the lower bound for the Shannon capacity for the vertex-transitive graphs, was proved by Sonnemann:

Theorem 13 (Sonnemann [4]). *If G is a vertex-transitive graph on n vertices, then for all graphs H,*

$$\alpha(G \boxtimes H) \leq \frac{|V(G)|}{|V(G')|}\alpha(G' \boxtimes H) \tag{11}$$

where G' is a subgraph of G.

If we substitute in the Theorem above for G' the maximum clique of G, we obtain the following bound for every vertex-transitive graph G and every graph H:

$$\alpha(G \boxtimes H) \leq \left\lfloor \frac{|V(G)|\alpha(H)}{\omega(G)} \right\rfloor \tag{12}$$

where $\lfloor x \rfloor$ denotes the floor function: the largest integer number no greater than x.

4. Kneser graphs

There exist classes of graphs with known exact value of Shannon capacity, for example, Kneser graphs (named after Martin Kneser, who first investigated them in 1955, see Fig. 4). It is known that Kneser graphs are both vertex-transitive and edge-transitive graphs.

Definition 14. *The Kneser graph $KG(n, r)$ is the graph whose vertices correspond to the r-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint.*

Clearly, Kneser graphs are non-empty iff $n \geq 2r$.

Theorem 15 (Erdős, Ko, Rado [5]). *Let $KG(n, r)$ be a Kneser graph with $n \geq 2r$. Then*

$$\alpha(KG(n, r)) = \binom{n-1}{r-1}. \tag{13}$$

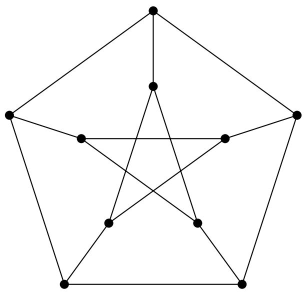


Figure 4: The Kneser graph $KG(5, 2)$, the so called Petersen graph

In fact, the eigenvalues of the adjacency matrix of Kneser graphs are known to be equal

$$(-1)^i \binom{q-r-i}{r-i} \quad \text{for } i = 0, 1, \ldots, r \tag{14}$$

This result, combined together with Theorem 15 was used to determine the Lovász number which turned out to give be equal to the Shannon capacity.

Theorem 16 (Lovász [2]).

$$C_0(KG(n, r)) = \binom{n-1}{r-1}. \tag{15}$$

5. Circular graphs

The class of vertex-transitive graphs contains many interesting subclasses. Particularly investigated in terms of Shannon capacity are so-called circular graphs.

Definition 17. *The circular graph* $C_n(1, \ldots, k)$ *is the graph whose vertex set is* $V = \{v_1, v_2, \ldots, v_n\}$ *and edge set* $E = \{v_i v_j : (i-j) \bmod n = 1 \vee \ldots \vee (i-j) \bmod n = k, i, j = 1, 2, \ldots, n\}$.

Brimkov et al. in [6] determined the Lovász function for $C_n(1, 2)$. Explicitly, this subclass of vertex-transitive graphs was first investigated by Baumert et al. [7], whose work was generalized in the following theorem by Badalyan and Markosyan:

Theorem 18 (**Badalyan, Markosyan [8]**). *Let $m, n \geq 3$ and $k, l \geq 1$. Then*

$$\alpha(C_m^l \boxtimes C_n^k) = \min\{\lfloor \alpha^*(C_m^l)\alpha(C_n^k)\rfloor, \lfloor \alpha(C_m^l)\alpha^*(C_n^k)\rfloor\} \tag{16}$$

In particular,

$$\alpha(C_n^k \boxtimes C_n^k) = \left\lfloor \frac{n}{k+1} \left\lfloor \frac{n}{k+1} \right\rfloor \right\rfloor \tag{17}$$

The upper bound, given in this theorem, follows from the inequality 12 and the well-known formula $\alpha(C_n^k) = \lfloor \frac{n}{k+1} \rfloor$.

We focused on the subgraph relation between circular graphs of different order and we proved the following theorem:

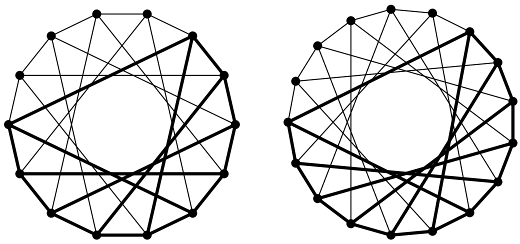
Theorem 19. *If $k \geq 2$ then both $C_{3k-1}(1,3)$ and $C_{3k-1}(1,k)$ contain $C_{2k}(1,k)$ as a subgraph.*

Proof. Let us denote by $v_1, v_2, \ldots, v_{3k-1}$ the consecutive vertices of the larger graph – and let us pick $2k$ vertices denoted by u_i ($1 \leq i \leq 2k$) from them. Let us also simplify the notation by substituting $v_{i \mod (3k-1)}$ by v_i and $u_{i \mod (3k-1)}$ by u_i – but keeping in mind that we oparate in field $\mathbb{Z}_{3k-1}$ and $\mathbb{Z}_{2k}$, respectively.

Then if we consider $C_{3k-1}(1,3)$, we may assign to each u_i its respective v_{3i} for every $i = 1, 2, \ldots 2k$. Then, every $u_i u_{i+1}$ edge corresponds to $v_{3i}v_{3i+3}$ so it is included in the original graph. Moreover, every $u_i u_{i+k}$ edge corresponds to $v_{3i}v_{3i+3k} = v_i v_{3i+3}$ therefore it is also included in the original graph. Finally, if $u_i u_j$ is an edge in the new graph then $v_{3i}v_{3j}$ is also an edge in the original graph so $3(i-j) \mod (3k-1) = 1$ or $3(i-j) \mod (3k-1) = 3$ which is possible only if either $i = j \pm k$ (first equation) or $i = j \pm 1$ (second equation) as $1 \leq i, j \leq 2k$ and $1 \leq |i-j| \mod (2k) \leq k$. Therefore, all vertices u_i form a circular graph $C_{2k}(1,k)$.

In case of $C_{3k-1}(1,k)$, the proof is much simpler: it suffices to notice that we may assign to each u_i its respective v_i for every $i = 1, 2, \ldots 2k$ and as in the previous case every vertex from $C_{2k}(1,k)$ is mapped on unique vertex from $C_{3k-1}(1,k)$, every $u_i u_{i+1}$ edge corresponds to $v_i v_{i+1}$ (if $i < 2k$) or to $v_{2k}v_{3k} = v_{2k}v_1$ (if $i = 2k$) and every $u_i u_{i+k}$ edge corresponds to $v_i v_{i+k}$ for every $1 \leq i \leq k$ and no $u_i u_{i+k}$ edge is included in the new graph if $k < i \leq 2k$. Clearly, by the same argument as before no other edges are included in the graph so all vertices u_i form a circular graph $C_{2k}(1,k)$. □

The examples of such graphs are presented in Fig. 5. Interestingly, for all circular graphs the possible circular subgraphs are always 3-regular (e.g. they are



(a) $C_{10}(1,5) \subset C_{14}(1,5)$ (b) $C_{12}(1,6) \subset C_{17}(1,6)$

Figure 5: The circular graphs $C_{14}(1,5)$ and $C_{17}(1,6)$ with theirs circular subgraphs

isomorphic to $C_{2k}(1,k)$ for some $k \geq 2$). This relation together with well known branch-and-bound or tabu search algorithms based on the Sonnemann bound (from the inequality 11) can be used to determine the values of the independence number for the squares of the circular graphs.

Jurkiewicz et al. [9] found partial results of the independence number for the third power of circular graphs $C_n(1,\ldots,k)$, presented in Table 1. Earlier known values ([7, 10]) are in the first column of the table. We can also find these values in the third column of Table 2, which presents similar values for cycles.

Baumert et al. [7] also showed that for all natural p and $n > 2$

$$\alpha(C_{n+2}^{\boxtimes p}) \geq 1 + \alpha(C_n^{\boxtimes p})\left(\frac{(n+2)^p - 2^p}{n^p}\right) \tag{18}$$

In addition, Codenotti et al. [10] obtained

$$\alpha(C_{n+i}^{\boxtimes p}) \geq k\frac{(n+1)^p - i^p}{n} + \left(\frac{i}{2}\right)^p \tag{19}$$

for each n of the form $n = k2^p + 1$, with $k > 0$ and $p > 1$.

From the inequality 5 and above facts (values from tables, inequalities 18 and 19), we can easily establish lower bounds of the Shannon capacity C_0 for considered graphs. For example

$$C_0(C_8(1,2)) \geq 2.2894 \tag{20}$$

$n \setminus k$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4	8	1	1	1	1	1	1	1	1	1
5	10	1	1	1	1	1	1	1	1	1
6	27	8	1	1	1	1	1	1	1	1
7	33	8	1	1	1	1	1	1	1	1
8	64	12	8	1	1	1	1	1	1	1
9	81	27	8	1	1	1	1	1	1	1
10	125	30	10	8	1	1	1	1	1	1
11	148	40 36	13	8	1	1	1	1	1	1
12	216	64	27	8	8	1	1	1	1	1
13	247	73 69	29 27	10	8	1	1	1	1	1
14	343	84 79	33	14	8	8	1	1	1	1
15	390 382	125	41 36	27	10	8	1	1	1	1
16	512	138 133	64	28 27	12	8	8	1	1	1
17	578	158 149	70 64	33 30	14	9	8	1	1	1
18	729	216	81	36	27	10	8	8	1	1
19	807	240 224	90 82	41 36	28 27	10	9	8	1	1
20	1000	266 247	125	64	30	14	10	8	8	1

Table 1: The exact values and known bounds for $\alpha(C_n(1,\ldots,k)^{\boxtimes 3})$

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n/p	**1**	**2**	**3**	**4**	**5**	**6**	p'	Lower bound on C_0
5	2	5	10	25	50	125	2	2.2361
7	3	10	33	115 108	402 343	1101	4	3.2237
9	4	18	81	363 324	1458	6561	3	4.3267
11	5	27	148	814 761	3996	21904	3	5.2896
13	6	39	247	1531	9633	61009	3	6.2743
15	7	52	390 382	2770	19864	145924	3	7.2558
17	8	68	578	4913	39304	334084	4	8.3721
19	9	85	807	7666	68994	651610	4	9.3571
21	10	105	1092	11441	114660	1201305	4	10.3423
23	11	126	1437	16466	181126	2074716	4	11.3278
25	12	150	1875	23125	281250	3515625	4	12.3316
27	13	175	2362	31522	413350	5579044	4	13.3246
29	14	203	2929	42017	594587	8579041	4	14.3171

Table 2: The exact values and known bounds for $\alpha((C_n)^{\boxtimes p})$

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