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EQUITABLE COLORINGS OF CORONA MULTIPRODUCTS OF GRAPHS

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Abstract

A graph is equitably k-colorable if its vertices can be partitioned into k independent sets in such a way that the numbers of vertices in any two sets differ by at most one. The smallest k for which such a coloring exists is known as the *equitable chromatic number* of G and denoted by $\chi_=(G)$. It is known that the problem of computation of $\chi_=(G)$ is NP-hard in general and remains so for corona graphs. In this paper we consider the same model of coloring in the case of corona multiproducts of graphs. In particular, we obtain some results regarding the equitable chromatic number for the l-corona product $G \circ^l H$, where G is an equitably 3- or 4-colorable graph and H is an r-partite graph, a cycle or a complete graph. Our proofs are mostly constructive in that they lead to polynomial algorithms for equitable coloring of such graph products provided that there is given an equitable

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coloring of G. Moreover, we confirm the Equitable Coloring Conjecture for corona products of such graphs. This paper extends our results from [7].

Keywords: corona graph, equitable chromatic number, equitable coloring conjecture, equitable graph coloring, multiproduct of graphs, NP-completeness, polynomial algorithm.

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1. Introduction

All graphs considered in this paper are finite, connected and simple, i.e. undirected, loopless and without multiple edges.

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, ..., V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable.* In the case, where each color is used the same number of times, i.e. $|V_i| = |V_j|$ for every pair (i, j), graph G is said to be *strongly equitably k-colorable.* The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* of G and denoted by $\chi_=(G)$. Since equitable coloring is a proper coloring with additional condition, the inequality $\chi(G) \leq \chi_=(G)$ holds for any graph G. It turns out that if a graph G has an equitable k-coloring, then it does not mean that it has also an equitable $(k+1)$-coloring. For example, $K_{3,3}$ admits equitable 2-coloring, but it is not equitably 3-colorable.

In some discrete industrial systems we can encounter the problem of partitioning a system with binary conflict relations into balanced conflict-free subsystems. Such situations can be clearly modeled by means of the equitable graph coloring. For example, equitable coloring algorithms can be used in scheduling and timetabling problems [6, 9].

The notion of equitable colorability was introduced by Meyer [15]. However, an earlier work of Hajnal and Szemerédi [10] showed that a graph G with maximal degree Δ is equitably k-colorable if $k \geq \Delta+1$. Recently, Kierstead et al. [11] have given an $O(\Delta|V(G)|^2)$-time algorithm for equitable $(\Delta+1)$-coloring of graph G. In his seminal paper, Meyer [15] formulated the following conjecture:

Conjecture 1 Equitable Coloring Conjecture (ECC)**.** *For any connected graph G with maximum degree Δ and other than a complete graph or an odd cycle, $\chi_=(G) \leq \Delta$.*

Chen, Lih and Wu made a stronger conjecture:

Conjecture 2 Equitable Δ-Coloring Conjecture, [3]**.** *If G is a connected graph of maximum degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any $n \geq 1$, then G is equitably Δ-colorable.*

Conjecture 1 has been verified for all graphs on six or fewer vertices. Lih and Wu [13] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [17] considered a broader class of graphs, namely r-partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. Conjecture 2 was confirmed for outerplanar graphs [18], series-parallel graphs [20], and planar graphs with maximum degree at least 9 [16, 19]. For the survey see [12].

In this paper we consider the same model of coloring in the case of corona products of graphs. The *corona* of two graphs, n-vertex graph G and m-vertex graph H, is a graph $G \circ H$ formed from one copy of G and n copies of H where the ith vertex of G is adjacent to every vertex in the ith copy of H. For any integer $l \geq 2$, we define the graph $G \circ^l H$ recursively from $G \circ H$ as $G \circ^l H = (G \circ^{l-1} H) \circ H$ (cf. Fig. 1). Graph $G \circ^l H$ is also named as *l-corona product* of G and H. Such type of graph product was introduced by Frucht and Harary [4].

The topic of equitable coloring was widely discussed in the literature. It was considered for some particular graph classes and also for several graph products: Cartesian [14], tensor [6], and coronas [7, 8]. The complexity of many problems, including equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complex prime factors. In adition to this, corona graphs lie close to the boundary between easy and NP-hard coloring problems [8].

A straightforward reduction from graph coloring to equitable coloring by adding sufficiently many isolated vertices to a graph, proves that it is NP-complete to test whether a graph has an equitable coloring with a given number of colors (greater than two). Furmańczyk and Kubale proved that the problem remains NP-complete for cubical coronas [8]. In this way they pointed out a class of graphs for which equitable coloring is harder than ordinary coloring. Bodlaender and Fomin [1] showed that the equitable coloring problem can be solved to optimality in polynomial time for graphs with bounded treewidth. Polynomial time algorithms are known for equitable coloring of split graphs [2], cubic graphs [8], and some coronas [7].

The remainder of the paper is organized as follows. In Section 2 we give an upper bound on the equitable chromatic number of l-corona product of graphs with complete graphs while in Section 3 we give some results concerning the equitable colorability of l-corona products of some graphs versus r-partite graphs. Next, in Section 4 we consider l-corona products of graphs G with $\chi_=(G) \leq 4$ and cycles. Section 5 summarizes our results in a tabular form. In this way we extend the class of graphs that can be colored optimally in polynomial time and confirm the ECC conjecture for the extended class of graphs.

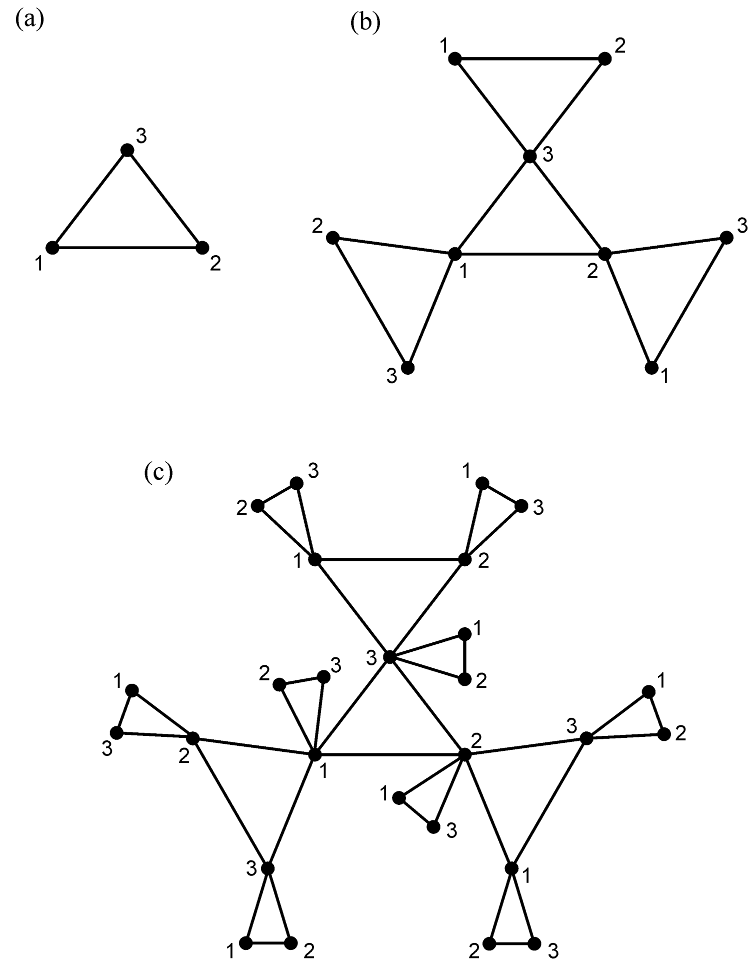

Figure 1. Example of graphs: a) C_3; b) $C_3 \circ K_2$; c) $C_3 \circ^2 K_2$.

2. Equitable coloring of corona multiproducts with complete graphs

It is known that $\chi_=(G \circ K_m) = m+1$ for every graph G such that $\chi(G) \leq m+1$ [7]. As $G \circ K_m$ is $(m+1)$-colorable, the graph $G \circ^2 K_m$ is also equitably $(m+1)$-colorable, and so on. Therefore, this result can be easy generalized to the l-corona product, $l \geq 1$.

Proposition 3. *If G is a graph with $\chi(G) \leq m+1$, then $\chi_=(G \circ^l K_m) = m+1$ for any $l \geq 1$.*

Let us note that since G is connected, the maximum degree of the corona $\Delta(G \circ^l K_m)$ is equal to $\Delta(G) + m \cdot l$. Since $m+1 \leq \Delta(G) + m \cdot l$, the ECC conjecture is true for such graphs.

Let us also notice that we immediately get an upper bound on the equitable chromatic number:

$$\chi_=(G \circ^l H) \leq m+1,$$

where $l \geq 1$, $\chi(G) \leq m+1$ and graph H is of order m.

3. EQUITABLE COLORING OF CORONA GRAPHS WITH r-PARTITE GRAPHS

In this section we consider corona products of a graph G and r-partite graphs, where G fulfills some additional conditions.

Theorem 4. *Let G be an equitably k-colorable graph on n vertices and let H be a $(k-1)$-partite graph. If k divides n (in symbols $k|n$), then for any $l \geq 1$ graph $G \circ^l H$ is equitably k-colorable.*

Proof. The proof is by induction on l.

Step 1: For $l = 1$ the theorem holds due to the following.

Suppose $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $V_1, \ldots, V_k$ are independent sets each of size n/k. This means that they form a strongly equitable k-coloring of G. For each vertex $z \in V(G)$, let $H_z = (X_1^z, \ldots, X_{k-1}^z, E^z)$ be the copy of $(k-1)$-partite graph $H = (X_1, \ldots, X_{k-1}, E)$ in $G \circ H$ corresponding to z. Let

$$\begin{aligned}
&V_1' = V_1 \cup \textstyle\bigcup_{z \in V_2} X_1^z \cup \cdots \cup \bigcup_{z \in V_k} X_{k-1}^z,\\
&V_2' = V_2 \cup \textstyle\bigcup_{z \in V_3} X_1^z \cup \cdots \cup \bigcup_{z \in V_k} X_{k-2}^z \cup \bigcup_{z \in V_1} X_{k-1}^z,\\
&\vdots\\
&V_{k-1}' = V_{k-1} \cup \textstyle\bigcup_{z \in V_k} X_1^z \cup \bigcup_{z \in V_1} X_2^z \cup \cdots \cup \bigcup_{z \in V_{k-2}} X_{k-1}^z,\\
&V_k' = V_k \cup \textstyle\bigcup_{z \in V_1} X_1^z \cup \cdots \cup \bigcup_{z \in V_{k-1}} X_{k-1}^z.
\end{aligned}$$

It is easy to see that $V(G \circ H) = V_1' \cup \cdots \cup V_k'$ is an equitable k-coloring of $G \circ H$. In this coloring each of the k colors is used exactly $n(1 + |X_1| + \cdots + |X_{k-1}|)/k$ times.

Step 2: Suppose Theorem 4 holds for some $l \geq 1$.

Step 3: We have to show that $(G \circ^l H) \circ H$ is equitably k-colorable. Let us note that if $k|n$ then the cardinality of vertex set of $G \circ^l H$, which is equal to $n(m+1)^l$, is also divisible by k. So using the inductive hypothesis we get immediately the conclusion.

■

Since any r-partite graph, where $r \leq k-1$, is also $(k-1)$-partite we have immediately

Corollary 5. *Let G be an equitably k-colorable graph on n vertices and let H be an r-partite graph with $r \leq k-1$. If $k|n$, then for any $l \geq 1$*

$$\chi_=(G \circ^l H) \leq k.$$

If G is an equitably 3-colorable graph on n vertices, and H is a bipartite graph, then Corollary 5 ensures that all corona multiproducts of G and H are equitably 3-colorable provided that $3|n$. One may wonder whether this result can be extended to the case when $3 \nmid n$. The theorem proved below gives a negative answer to this question.

Theorem 6. *Let G be an equitably 3-colorable graph on n vertices, and assume that $3 \nmid n$. Moreover, let H be a connected bipartite graph with equal size of partitions and $|V(H)| \geq 6$. Then multicorona products $G \circ^l H$ are not equitably 3-colorable for $l \geq 1$.*

Proof. We first observe that if $G \circ^l H$ is 3-colored, then the colors on the vertices of G uniquely determine the colors used on each copy of H. Moreover, since H is connected, each copy of H must be equitably 2-colored. Hence, if we assume that $G \circ^l H$ is equitably 3-colored, and adopt the convention that $G \circ^0 H = G$, we can write the following equalities for the number of vertices with colors 1, 2 and 3, respectively:

$$\begin{aligned}
|V_1(G \circ^l H)| &= |V_1(G \circ^{l-1} H)| + (|V_2(G \circ^{l-1} H)| + |V_3(G \circ^{l-1} H)|) \cdot \tfrac{|V(H)|}{2}, \\
|V_2(G \circ^l H)| &= |V_2(G \circ^{l-1} H)| + (|V_1(G \circ^{l-1} H)| + |V_3(G \circ^{l-1} H)|) \cdot \tfrac{|V(H)|}{2}, \\
|V_3(G \circ^l H)| &= |V_3(G \circ^{l-1} H)| + (|V_1(G \circ^{l-1} H)| + |V_2(G \circ^{l-1} H)|) \cdot \tfrac{|V(H)|}{2}.
\end{aligned}$$

Here V_k, for $k = 1, 2, 3$, denotes the set of vertices of the corresponding graph with color k. Define:

$$m_l = \max_{1 \leq i < j \leq 3} \left| |V_i(G \circ^l H)| - |V_j(G \circ^l H)| \right|.$$

Then, for $l \geq 1$

$$m_l = m_{l-1} \cdot \left(\frac{|V(H)|}{2} - 1\right).$$

Since $3 \nmid n$, we have $m_0 \geq 1$. Taking into account that $|V(H)| \geq 6$, we get

$$m_l = m_0 \cdot \left(\frac{|V(H)|}{2} - 1\right)^l \geq 2^l \geq 2,$$

which means that the coloring is not equitable, contradicting our assumption. The proof of Theorem 6 is completed. ■

If G is an equitably 4-colorable graph on n vertices, and H is a bipartite graph, then Corollary 5 implies that all corona multiproducts of G and H are equitably 4-colorable provided that $4|n$. One may wonder whether this result can be extended to the case when $4 \nmid n$. Below, we will obtain a result that gives a partial answer to this question.

We will need the following lemma.

Lemma 7. *Let G be a graph on 4 vertices, and let H be a bipartite graph with bipartition A and B, such that $|A| = |B|$ and $|V(H)|$ is divisible by 4. Then $G \circ H$ admits an equitable 4-coloring such that the vertices of G have pairwise different colors.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices of G. Let H_1, H_2, H_3 and H_4 be the copies of H corresponding to these vertices. Moreover, for $i = 1, 2, 3, 4$ let A_i and B_i be the bipartition sets of H_i. Consider a coloring of vertices of $G \circ H$ obtained as follows: color v_1, v_2, v_3, v_4 with colors 1, 2, 3 and 4, respectively, color the vertices of A_1 and A_4 with color 2, color the vertices of B_1 and B_4 with color 3, color the vertices of A_2 and A_3 with color 1, and color the vertices of B_2 and B_3 with color 4. One can easily verify that this is an equitable 4-coloring of $G \circ H$ meeting the requirements of the lemma. ■

Now, we prove our theorem on equitable 4-colorings.

Theorem 8. *Let G be an equitably 4-colorable graph on $n \geq 2$ vertices, and let H be a bipartite graph with bipartition A and B, such that $|A| = |B| = m/2$ and m is divisible by 4. Then multicorona products $G \circ^l H$ are equitably 4-colorable for $l \geq 1$.*

Proof. We first observe that it suffices to prove the theorem for $l = 1$. The rest follows from an induction on l. Thus, we will only show that $G \circ H$ is equitably 4-colorable.

Consider an equitable 4-coloring of G. Let $t \equiv n (\mathrm{mod} 4)$. We will consider 4 cases.

Case 1: $t = 0$

In this case $4|n$, hence the equitable 4-coloring of G colors the vertices of G with colors 1, 2, 3 and 4 so that the color classes are of the same cardinality. Partition the vertices of G into $n/4$ groups $V_1, \ldots, V_{n/4}$, so that each group contains 4 vertices of different colors. For $j = 1, \ldots, n/4$ the graphs $G[V_j] \circ H$ are equitably 4-colorable due to Lemma 7, where $G[V_j]$ is a subgraph of G induced by V_j. Since m is divisible by 4, this results in an equitable 4-coloring of $G \circ H$.

Case 2: $t = 2$

In this case $n \equiv 2 (\mathrm{mod} 4)$. Hence, without loss of generality, we can assume that in the equitable 4-coloring of G, the vertices with colors 1 and 2 contain one more vertex than the vertices with colors 3 and 4. Let v_1 and v_2 be two vertices with colors 1 and 2, respectively. Similarly to Case 1, one can show that $(G - \{v_1, v_2\}) \circ H$ is equitably 4-colorable. Observe that the color classes of this graph are going to be of equal size.

Now, we show that $G[\{v_1, v_2\}] \circ H$ is equitably 4-colorable. Let H_1 and H_2 be the copies of H corresponding to v_1 and v_2, respectively. Moreover, for $i = 1, 2$ let A_i and B_i be the bipartition of H_i. Color the vertices of A_1 with color 2, vertices of B_1 with color 3, vertices of A_2 with color 4, and vertices of B_2 with color 1, respectively. One can easily check that this results in an equitable 4-coloring of $G \circ H$.

Case 3: $t = 3$

In this case $n \equiv 3 (\mathrm{mod} 4)$. Hence, without loss of generality, we can assume that in the equitable 4-coloring of G, the vertices with colors 1, 2 and 3 contain one more vertex than the vertices with color 4. Let v_1, v_2 and v_3 be three vertices with colors 1, 2 and 3, respectively. Similarly to Case 1, one can show that $(G - \{v_1, v_2, v_3\}) \circ H$ is equitably 4-colorable. Observe that the color classes of this graph are going to be of equal size.

Now, we show that $G[\{v_1, v_2, v_3\}] \circ H$ is equitably 4-colorable. Let H_1, H_2 and H_3 be the copies of H corresponding to v_1, v_2 and v_3, respectively. Moreover, for $i = 1, 2, 3$ let A_i and B_i be the bipartition of H_i. Color the vertices of A_1 with color 4, half of vertices of B_1 with color 2 and the other half with color 3, vertices of A_2 with color 1, and vertices of B_2 with color 3, half of vertices of A_3 with color 1 and the other half with color 4, and vertices of B_4 with color 2, respectively. One can easily check that this results in an equitable 4-coloring of $G \circ H$.

Case 4: $t = 1$

In this case $n \equiv 1 (\mathrm{mod} 4)$. Hence, without loss of generality, we can assume that in the equitable 4-coloring of G, the vertices with color 1 contain one more vertex than the vertices with colors 2, 3 and 4. Let v_1, v_2, v_3, v_4 and v_5 be five vertices with colors 1, 2, 3 and 4, such that v_i is of color i for $i = 1, 2, 3, 4$, and v_5 is of color 1. Observe that we can always choose such five vertices, since $n \geq 2$. Similarly to Case 1, one can show that $(G - \{v_1, v_2, v_3, v_4, v_5\}) \circ H$ is equitably 4-colorable. Observe that the color classes of this graph are going to be of equal size.

Now, we show that $G[\{v_1, v_2, v_3, v_4, v_5\}] \circ H$ is equitably 4-colorable. Let H_1, H_2, H_3, H_4 and H_5 be the copies of H corresponding to v_1, v_2, v_3, v_4 and v_5, respectively. Moreover, let A_i and B_i be the bipartition sets of H_i, $i = 1, \ldots, 5$.

Color the vertices of A_1 with color 2, the vertices of B_1 with color 3, the vertices of A_2 with color 4, the vertices of B_2 with color 1, the vertices of A_3 with color 1, the vertices of B_3 with color 4, half of vertices of A_4 with color 3 and the other half with color 1, vertices of B_4 with color 2, half of vertices of A_5 with color 2 and the other half with color 4, and the vertices of B_5 with color 3, respectively. One can easily check that this results in an equitable 4-coloring of $G \circ H$.

The proof of the theorem is completed. ■

4. EQUITABLE COLORING OF CORONA MULTIPRODUCTS WITH CYCLES

In this section we consider corona products of a graph G and cycles. We will consider two main cases depending on the parity of m.

Theorem 9. *Let G be an equitably 3-colorable graph on $n \geq 1$ vertices and let m be even. If $3|n$ or $m = 4$, then*

$$\chi_=(G \circ^l C_m) = 3$$

for each $l \geq 1$.

Proof. Of course, we cannot use fewer than three colors, as $\chi(G \circ^l C_m) = 3$. The first part of the theorem, for $3|n$, follows from Corollary 5.

The case when $m = 4$ was partially considered in [7]. The authors proved that if G is an equitably 3-colorable graph on $n \geq 2$ vertices, then $\chi_=(G \circ C_4) = 3$. It is easy to see that also for $n = 1$ this equality holds, i.e. $\chi_=(K_1 \circ C_4) = 3$. This means that our theorem is true for $l = 1$. The remaining part of this proof is by induction on the number l, similar to that in the proof of Theorem 4. ■

We also know that in the remaining cases, i.e. when G is equitably 4-colorable or $3 \nmid n$, we need more than three colors for equitable coloring of $G \circ C_m$, even if m is even [7].

Theorem 10. *If G is equitably 4-colorable and $l \geq 2$, then the graph $G \circ^l C_m$ is equitably 4-colorable for each even $m \geq 4$.*

Proof. Let us consider two cases.

Case 1: $4|n$

The conclusion follows immediately from Theorem 4.

Case 2: $4 \nmid n$

First, we will show that our theorem is true for $l = 2$ and then by induction on l we will get the conclusion for multicoronas $G \circ^l C_m$, $l \geq 2$.

Step 1: $l = 2$

- $n = 1$
  Now, we have to prove that there is an equitable 4-coloring of $K_1 \circ^2 C_m$. First, we color with 1 the vertex of K_1, next the vertices of C_m in $K_1 \circ C_m$ with colors 2 and 3 using each of them $m/2$ times. Next, we color appropriately the vertices in one copy linked to vertex colored with 1, $m/2$ copies linked to vertices colored with 2, and $m/2 - 1$ copies linked to vertices colored with 3 using each time two of three allowed colors. In particular, we use color $i - 1$ and $i + 1$ in the copy linked to vertex colored with i (operations are applied modulo 4). One copy of C_m remains still uncolored. We color the vertices of it properly with color 1 and 0. In such a coloring color 1 is used $(m/2 + 1)m/2 + 1$ times, while any other color is used $(m/2 + 1)m/2$ times. Thus the coloring is equitable for each even $m \geq 4$.
- $n \geq 2$
  First, we color G equitably with 4 colors and arrange the cardinalities of color classes in a non-increasing order. Next, we renumber the vertices of G so that for each $i = 1, \ldots, n$ vertex v_i has color $i \bmod 4$. After that we color $G \circ C_m$ using for the copy adjacent to each v_i $m/2$ times color $(i \bmod 4+1) \bmod 4$, $\lceil m/4 \rceil$ times color $(i \bmod 4 + 2) \bmod 4$ and $\lfloor m/4 \rfloor$ times color $(i \bmod 4 + 3) \bmod 4$. Note that this coloring is not equitable. Therefore we have to recolor some of the copies of C_m. To this aim we consider three subcases.
  (i) $n \equiv 1 (\mathrm{mod} 4)$
  In this case we recolor:
  - the copy linked to v_1 using $m/2$ times color 3, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 2.

- the copy linked to v_2 using $m/2$ times color 1, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 3,
- the copy linked to v_3 using $m/2$ times color 1, and $m/2$ times color 0,
- the copy linked to v_4 using $m/2$ times color 2, $\lceil m/4 \rceil$ times color 3, and $\lfloor m/4 \rfloor$ times color 1.

(ii) $n \equiv 2(\text{mod} 4)$
In this case we recolor the copy linked to v_2 using $m/2$ times color 1, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 3.

(iii) $n \equiv 3(\text{mod} 4)$
In this case we recolor:

- the copy linked to v_1 using $m/2$ times color 3, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 2,
- the copy linked to v_2 using $m/2$ times color 1, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 3,
- the copy linked to v_3 using $m/2$ times color 2, $\lceil m/4 \rceil$ times color 0, and $\lfloor m/4 \rfloor$ times color 1.

One can easily check that, in each subcase, the obtained coloring is an equitable 4-coloring with colors $\{0, 1, 2, 3\}$. Let $G' = G \circ C_m$. Now, we repeat the above procedure to get an equitable 4-coloring of $G' \circ C_m = G \circ^2 C_m$.

Step 2: *Induction hypothesis for some $l \geq 2$.*

Step 3: *The proof that $G \circ^{l+1} C_m$ is equitably 4-colorable.*
Since $G \circ^{l+1} C_m = (G \circ^l C_m) \circ C_m$ and the fact that we have an equitable 4-coloring of the center graph $G \circ^l C_m$ by the induction hypothesis, we can extend the coloring into an equitable 4-coloring of $G \circ^{l+1} C_m$ in the way described above.

■

It turns out that in the case when the number of vertices of graph G is not divisible by three, the weak inequality becomes equality.

Theorem 11. *Let G be an equitably 3- or 4-colorable graph on $n \geq 2$ vertices and let $l \geq 1$. If $3 \nmid n$, then*

$$\chi_=(G \circ^l C_m) = 4$$

for each even $m \geq 6$.

Proof. Due to Theorem 10 all we need is the proof that we cannot use fewer colors.

If $\chi(G) = 4$ then of course $\chi_=(G \circ^l C_m) = 4$, for any l. Let us assume that $\chi(G) \leq 3$. Note that any 3-coloring of G uniquely determines a 3-coloring of $G \circ^l C_m$. When we color the vertices in a copy of C_m linked to a vertex of $G \circ^{l-1} C_m$, we use two available colors. It is not hard to notice that the difference between cardinalities of color classes is the smallest when 3-coloring of G is strongly equitable. In our case, since n is not divisible by three, a strongly equitable coloring does not exist. If the maximal difference between cardinalities of any two color classes of G is 1, any 3-coloring of $G \circ^l C_m$ cannot be equitable. This follows from the following reasoning.

We claim that every equitable (not strongly) 3-coloring of G determines a 3-coloring of $G \circ^l C_m$ with maximum difference among the color classes greater than 1. Indeed, for $l = 1$ we have:

(i) $n \equiv 1(\mathrm{mod} 3)$

Cardinalities of color classes for colors 1, 2 and 3 are equal to $\lfloor n/3 \rfloor (m + 1) + 1$, $\lfloor n/3 \rfloor (m + 1) + m/2$ and $\lfloor n/3 \rfloor (m + 1) + m/2$, respectively. The maximum difference between color classes is equal to $m/2 - 1 \geq 2$.

(ii) $n \equiv 2(\mathrm{mod} 3)$

Cardinalities of color classes for colors 1, 2 and 3 are equal to $\lfloor n/3 \rfloor (m + 1) + 1 + m/2$, $\lfloor n/3 \rfloor (m + 1) + 1 + m/2$ and $\lfloor n/3 \rfloor (m + 1) + m$, respectively. The maximum difference between color classes is equal to $m/2 - 1 \geq 2$.

The reader may verify that the maximal difference between the cardinalities of color classes in multicorona $G \circ^l C_m$ is $(m/2 - 1)^l$, which is growing as l tends to infinity.

■

Now, we consider cycles on odd number of vertices. First, let us recall a result for coronas $G \circ C_m$, where m is odd.

Theorem 12 [7]**.** *If G is equitably 4-colorable graph on $n \geq 2$ vertices and $m \geq 3$ is odd, then*

$$\chi_=(G \circ C_m) = 4.$$

Now, we generalize this result to multicoronas.

Theorem 13. *If G is an equitably 4-colorable graph on $n \geq 2$ vertices and $l \geq 1$, then*

$$\chi_=\left(G \circ^l C_m\right) = 4$$

for each odd $m \geq 3$.

Proof. We have $\chi_=(G \circ^l C_m) \geq 4$, since $G \circ^l C_m$ contains $K_1 \circ C_m$ as a subgraph. On the other hand, we get the inequality $\chi_=(G \circ^l C_m) \leq 4$ by starting from an initial induction step based on Theorem 12 and applying a similar inductive argument to that used in the proof of Theorem 10. ■

We have considered equitable coloring of corona product of graphs on at least one vertex and even cycles or corona of graphs on at least two vertices and odd cycles. Now, for the sake of completeness, we consider equitable colorings of corona products of one isolated vertex and odd cycles. It is easy to see that

$$\chi_=(K_1 \circ C_m) = \begin{cases} 4, & \text{if } m = 3, \\ \lceil \frac{m}{2} \rceil + 1, & \text{if } m > 3. \end{cases} \tag{1}$$

Though the value of equitable chromatic number of multicorona $K_1 \circ^l C_m$ can be arbitrarily large for $l = 1$, the situation changes significantly for larger values of l.

Theorem 14. *If $m \geq 3$, $l \geq 2$, then*

$$\chi_=(K_1 \circ^l C_m) = \begin{cases} 3, & \textit{if } m = 4, \\ 4, & \textit{otherwise.} \end{cases}$$

Proof. We have to consider three cases:

Case 1: *m is even*

First of all, we have $\chi_=(K_1 \circ^l C_m) \leq 4$ due to Theorem 10. Next, observe that a 3-coloring of $K_1 \circ^l C_m$ is uniquely determined up to permutations of colors. This coloring is equitable only for $m = 4$.

Case 2: $m = 3$

Since $C_3 = K_3$, our conclusion follows immediately from Proposition 3.

Case 3: *m is odd and $m \geq 5$*

Observe that at least 4 colors are necessary, since $K_1 \circ^2 C_m$ includes $K_1 \circ C_m$ as a subgraph. Below we present an equitable coloring with 4 colors.

Our proof is by induction on l.

Step 1: For $l = 2$ the theorem holds due to the following.
Let us notice that $|V(K_1 \circ^2 C_m)| = (m+1)^2$. This means that each of four colors must be used exactly $(m+1)^2/4 = (\lfloor m/2 \rfloor + 1)^2$ times in every equitable coloring. The graph $K_1 \circ^2 C_m$ consists of $m+1$ copies of C_m joined to vertices of $K_1 \circ C_m$ appropriately. The equitable 4-coloring of $K_1 \circ^2 C_m$ is formed as follows:

- the vertex of K_1 is colored with 1
- the remaining vertices of $K_1 \circ C_m$ are assigned colors 2, 3 and 4 with cardinalities equal to $\lfloor m/2 \rfloor$, $\lfloor m/2 \rfloor$ and 1, respectively
- the copy of C_m in $K_1 \circ^2 C_m$ joined to vertex colored 1 is assigned colors 2, 3 and 4 with cardinalities equal to 1, $\lfloor m/2 \rfloor$ and $\lfloor m/2 \rfloor$, respectively
- copies of C_m in $K_1 \circ^2 C_m$ joined to vertex colored 2 are assigned colors 1, 3 and 4 with cardinalities in each cycle equal to 1, $\lfloor m/2 \rfloor$ and $\lfloor m/2 \rfloor$, respectively
- copies of C_m in $K_1 \circ^2 C_m$ joined to vertex colored 3 are assigned colors 1, 2 and 4 with cardinalities in each cycle equal to $\lfloor m/2 \rfloor$, $\lfloor m/2 \rfloor$ and 1, respectively
- copies of C_m in $K_1 \circ^2 C_m$ joined to vertex colored 4 are assigned colors 1, 2 and 3 with cardinalities in each cycle C_m equal to $\lfloor m/2 \rfloor$, $\lfloor m/2 \rfloor$ and 1, respectively.

In such a coloring each of 4 colors is used exactly $(m+1)^2/4$ times.

Step 2: *Induction hypothesis.* Suppose Theorem 14 holds for some $l \geq 2$.

Step 3: We have to show that $\chi_=((K_1 \circ^l C_m) \circ C_m) = 4$. The conclusion follows from Theorem 13.

∎

Since $G \circ^l P_m$ is a subgraph of $G \circ^l C_m$, we have similar bounds on equitable chromatic number of coronas of appropriate graph G and a path as it was in the case of $G \circ^l C_m$, namely

$$\chi_=(G \circ^l P_m) \leq \chi_=(G \circ^l C_m).$$

5. Conclusion

In the paper we have given some results concerning multicorona products of low chromaticity graphs (bipartite, cycles, etc.) that confirm the Equitable Coloring Conjecture. In particular, we have shown that the ECC conjecture follows for every l-corona product $G \circ^l H$, where graph H is on m vertices and graph G is on n vertices and can be properly colored with $m-1$ colors. Moreover, we have established some special cases of products $G \circ^l H$ that can be colored with 3 or 4 colors efficiently provided that an equitable coloring of G can be done in polynomial time $p(n)$. This is in sharp contrast to cubical coronas for which equitable coloring with 4 colors is NP-hard [8]. The main of our results are summarized in Table 1.

<table>
<tr><td colspan="2" rowspan="2">H
G</td><td rowspan="2">bipartite graphs</td><td colspan="2">even cycles C_m</td><td rowspan="2">odd cycles</td></tr>
<tr><td>$m = 4$</td><td>$m \geq 6$</td></tr>
<tr><td rowspan="2">equitably 3-colorable graph G on $n \geq 2$ vertices</td><td>$3|n$</td><td>3 [Thm. 4]</td><td rowspan="2">3 [Thm. 9]</td><td>3 [Thm. 9]</td><td rowspan="2">4 [Thm. 13]</td></tr>
<tr><td>$3 \nmid n$</td><td>$\geq 4^*$ [Thm. 6]</td><td>4 [Thm. 11]</td></tr>
<tr><td rowspan="2">equitably 4-colorable graph G on $n \geq 2$ vertices</td><td>$3|n$</td><td rowspan="2">$\leq 4^{**}$ [Thm. 8]</td><td rowspan="2">≤ 4 [Thm. 10]</td><td>≤ 4 [Thm. 10]</td><td rowspan="2">4 [Thm. 13]</td></tr>
<tr><td>$3 \nmid n$</td><td>4 [Thm. 11]</td></tr>
</table>

Table 1. Possible values of the equitable chromatic number of coronas $G \circ^l H$, $l \geq 2$. Asterix $(*)$ means that the result is valid for H being balanced connected bipartite with $m \geq 6$. Double asterix $(**)$ means that the result is valid for H being balanced bipartite with $4|m$.

Since the time spent on coloring/recoloring of any vertex of $G \circ^l H$ is constant, such a coloring of graphs under consideration can be done in time $O(p(n) + nm^{-1}(m+1)^{l+1})$, which is polynomial in the size of $G \circ^l H$. For example, the following graphs:

- broken spoke wheels [5],
- reels [5],
- cubic graphs except K_4 [8],
- some graph products [6, 14],
- some cubical coronas [8]

admit equitable 3-coloring in polynomial time, and so do the corresponding multicoronas.

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