# Equitable and semi-equitable coloring of cubic graphs and its application in batch scheduling*

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**Abstract**

In the paper we consider the problems of equitable and semi-equitable coloring of vertices of cubic graphs. We show that in contrast to the equitable coloring, which is easy, the problem of semi-equitable coloring is NP- complete within a broad spectrum of graph parameters. This affects the complexity of batch scheduling of unit-length jobs with cubic incompatibility graph on three uniform processors to minimize the makespan.

**Keywords:** batch scheduling, equitable coloring, semi-equitable coloring, cubic graph

## 1 Introduction

Graph coloring belongs to the hardest combinatorial optimization problems. Therefore, some highly-structured graphs and simplified models of coloring are subject to

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consideration more often than not. Among others cubic, i.e. 3-regular, graphs are such highly-structured graphs (see Figs. 2, 3 and 4). These graphs lay close to the boundary between polynomial and NP-complete coloring problems. For example, the problem of coloring the vertices of cubic graphs is linear [7], while the problem of coloring the edges of such graphs is NP-hard.

We say that a graph $G = (V, E)$ is *equitably $k$-colorable* if and only if its vertex set can be partitioned into independent sets $V_1, \ldots, V_k \subset V$ such that $|V_i| - |V_j| \in \{-1, 0, 1\}$ for all $i, j = 1, \ldots, k$. The smallest $k$ for which $G$ admits such a coloring is called the *equitable chromatic number* of $G$ and denoted $\chi_=(G)$. For example, Chen et. al. [1] proved that every cubic graph can be equitably colored without adding a new color. Hence we have

$$\chi_=(G) = \chi(G) \tag{1}$$

for this class of graphs.

Graph $G$ has a *semi-equitable coloring*, if there exists a partition of its vertices into independent sets $V_1, \ldots, V_k \subset V$ such that one of these subsets, say $V_i$ is of size $s \notin \{\lfloor n/k \rfloor, \lceil n/k \rceil\}$, and the remaining subgraph $G - V_i$ is equitably $(k-1)$-colorable.

It is easy to see that straightforward reduction from classical graph coloring to equitable coloring by introducing sufficiently many isolated vertices to a graph proves that it is NP-complete to test whether a general graph has an equitable coloring with a given number of colors (greater than two). Thus in general the NP-hardness of equitable coloring implies the NP-hardness of classical coloring and vice versa. But is it true for any particular class of graphs? Or, in other words, is there a class of graphs for which the problem of equitable coloring is harder than the problem of classical coloring? Although it is relatively easy to give a class of graphs for which ordinary coloring is harder than equitable coloring (e.g. graphs with a spanning star), so far no class of graphs has been known showing the other way round, i.e. that equitable coloring is harder than ordinary one. However, the authors have given the positive answer to this question [3]. They found a class of cubical coronas for which

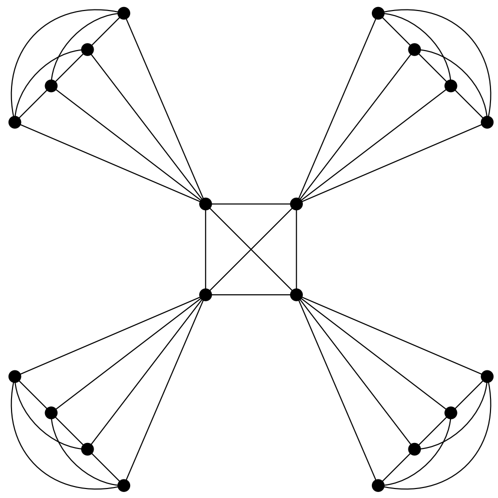

Figure 1: The smallest cubical corona $K_4 \circ K_4$.

the equitable 4-coloring is NP-hard, while the classical 4-coloring is polynomial. An example of cubical corona is shown in Fig. 1.

This model of coloring has many practical applications. Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring. One motivation for equitable graph coloring stems from scheduling problems. In this application the vertices of a graph represent a collection of jobs to be performed, and each edge connects two jobs that should not be performed at the same time. A coloring of this graph represents a partition of jobs into subsets that may be performed simultaneously. Due to load balancing considerations, it is desirable to perform equal or nearly-equal numbers of jobs in each time slot, and this balancing is exactly what an equitable coloring achieves. A good example of this type of scheduling is the problem of assigning university courses to time slots in a way that avoids incompatible pairs of courses at the same time and spreads the courses evenly among the available time slots, since then the usage of scarce additional resources (e.g. rooms) is maximized.

In this paper, we focus on the problem of equitable and semi-equitable coloring of cubic graphs. In particular, we show that in contrast to the equitable coloring problem for such graphs, which is easy, the semi-equitable coloring problem for cubic graphs becomes NP-hard. We than give a simple example of using this model of coloring in a problem of batch processing on three uniform batching machines to minimize the makespan (i.e. maximal batch completion time).

## 2 Tripartite cubic graphs

Note that if $G$ is a bipartite cubic graph then any 2-coloring is equitable and there may be no equitable 3-coloring (cf. $K_{3,3}$). On the other hand, all cubic graphs with $\chi(G) = 2$ have semi-equitable 3-coloring of type $(n/2, \lceil n/4 \rceil, \lfloor n/4 \rfloor)$. Moreover, they are easy colorable in linear time while traversing in a depth-first search (DFS) way. Therefore, in the sequel we shall concentrate on the hardest case, i.e. tripartite cubic graphs. We denote the class of 3-chromatic cubic graphs by $\mathcal{Q}_3$. Next, let $\mathcal{Q}_3(n) \subset \mathcal{Q}_3$ stand for the class of tripartite cubic graphs on $n$ vertices.

We consider the following combinatorial decision problems:

$\mathrm{IS}_3(Q, k)$: given a cubic graph $Q$ on $n$ vertices and an integer $k$, the question is: does $Q$ have an independent set $I$ of size at least $k$?

and its subproblem

$\mathrm{IS}_3(Q, .4n)$: this means the $\mathrm{IS}_3(Q, k)$ problem with $10|n$ and $k = 4n/10$.

Note that the $\mathrm{IS}_3(Q, k)$ problem is NP-complete [5] and remains so even if $n$ is a multiplication of 10. This is so because we can enlarge $Q$ by adding $j$ $(0 \le j \le 4)$ isolated copies of $K_{3,3}$ to it so that the number of vertices in the new graph is divisible by 10. Graph $Q$ has an independent set of size at least $k$ if and only if the new graph has an independent set of size at least $k + 3j$. Next theorems will imply considerable complexity problems as far as semi-equitable coloring is concerned.

**Theorem 1.** *Problem* $\mathrm{IS}_3(Q, .4n)$ *is* NP-*complete.*

*Proof.* Our polynomial reduction is from NP-complete problem $\mathrm{IS}_3(Q,k)$. For an $n$-vertex cubic graph $Q$ fulfilling $10|n$, and an integer $k$, let $r = |4n/10 - k|$. If $k \geq 4n/10$ then we construct a cubic graph $G = Q \cup rK_4 \cup rP$ else we construct $G = Q \cup rK_4 \cup 2rP \cup 4rK_{3,3}$, where $P \in Q_3(6)$ is the prism graph (see Fig. 2). It is easy to see that the answer to problem $\mathrm{IS}_3(Q,k)$ is 'yes' if and only if the answer to problem $\mathrm{IS}_3(G,.4n)$ is 'yes'. $\square$

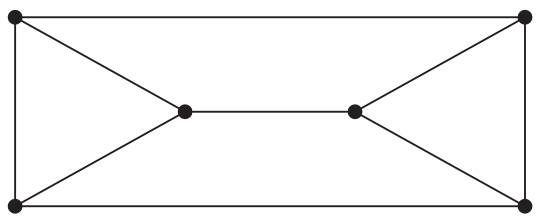

Figure 2: Example of a cubic graph - the prism graph $P$ on 6 vertices.

**Theorem 2.** *Let $Q \in \mathcal{Q}_3(n)$ and let $k = 4n/10$, where $10|n$. The problem of deciding whether $Q$ has a coloring of type $(4n/10, 3n/10, 3n/10)$ is* NP*-complete.*

*Proof.* We prove that $Q$ has a coloring of type $(4n/10, 3n/10, 3n/10)$ if and only if there is an affirmative answer to $\mathrm{IS}_3(Q,.4n)$.

Suppose first that $Q$ has the above 3-coloring. Then the color class of size $4n/10$ is an independent set that forms a solution to $\mathrm{IS}_3(Q,.4n)$.

Now suppose that there is a solution $I$ to $\mathrm{IS}_3(Q,.4n)$. Thus $|I| \geq 4n/10$. We know from [4] that in this case there exists an independent set $I'$ of size exactly $4n/10$ such that the subgraph $Q - I'$ is an equitably 2-colorable bipartite graph. This means that $Q$ can be 3-colored so that the color sequence is $(4n/10, 3n/10, 3n/10)$. $\square$

On the other hand, let us note that a cubic graph usually has such a big independent set. Frieze and Suen [2] have proven that for almost all cubic graphs $Q$ their independence number $\alpha(Q)$ fulfills the inequality $\alpha(Q) \geq 4.32n/10 - \epsilon n$ for any $\epsilon > 0$. In practice this means that a random graph from $\mathcal{Q}_3(n)$ is very likely to have an independent set of size $k \geq 4n/10$ and the probability of this fact increases with $n$.

Note that the existence of an independent set $I$ of size at least $4n/10$ does not mean that $Q - I$ is bipartite. It may happen that there remain some odd cycles in $Q - I$, even if a big independent set is found (see Fig. 3). Nevertheless, the authors proved in [4]

**Theorem 3.** *If $Q \in \mathcal{Q}_3(n)$ and $\alpha(Q) \geq 4n/10$, then there exists an independent set $I$ of size $k$ in $Q$ such that $Q - I$ is bipartite for $\lfloor (n - \alpha(Q))/2 \rfloor \leq k \leq \alpha(Q)$.* $\square$

The proof of this theorem is constructive, i.e. it gives an algorithm that transforms, step by step, an independent set $I'$ into an independent set $I$ such that $|I| = |I'|$ and $Q - I$ is bipartite. The complexity of this algorithm is polynomial.

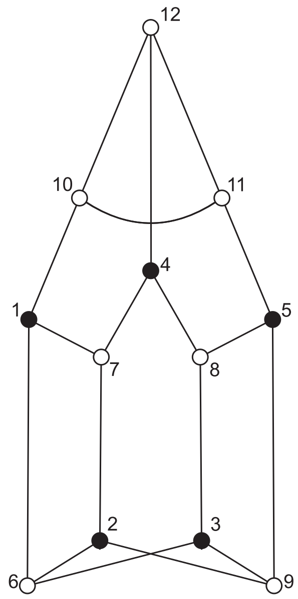


Figure 3: Example of a graph $Q$ for which a greedy algorithm, which repeatedly eliminates the closed neighborhood of a minimum degree vertex, finds an independent set $I$ (vertices in black) such that $Q - I$ contains $K_3$.

We state the question: If $|I| \geq 4n/10$ and $Q - I$ is bipartite, does $Q - I$ is equitably 2-colorable? The answer is affirmative. Indeed, assume that $|I| = 4n/10$. Notice that

$6n/10$ vertices of $Q - I$ induce binary trees (some of them may be trivial) and/or graphs whose 2-core is equibipartite (even cycle possibly with chords). Note that deleting an independent set $I$ of cardinality $4n/10$ from a cubic graph $Q$ means also that we remove $12n/10$ edges from the set of all $15n/10$ edges of $Q$. The resulting graph $Q - I$ has $6n/10$ vertices and $3n/10$ edges. Let $s_i$, $0 \leq i \leq 3$, be the number of vertices in $Q - I$ of degree $i$. Certainly, $s_0 + \ldots + s_3 = 6n/10$. Since the number of edges is half of the number of vertices, the number of isolated vertices, $s_0$, is equal to $s_2 + 2s_3$. If $s_0 = 0$, then $Q - I$ is a perfect matching and its equitable coloring is obvious. Suppose that $s_0 > 0$. Let $L$ denote the set of isolated vertices in $Q - I$. Let us consider subgraph $Q - I - L$. Each vertex of degree 3 causes the difference between cardinalities of color classes $\leq 2$, similarly each vertex of degree 2 causes the difference at most 1. The difference between the cardinalities of color classes in any coloring fulfilling these conditions does not exceed $s_2 + 2s_3$ in $Q - I - L$. Thus, the appropriate assignment of colors to isolated vertices in $L$ makes the whole graph $Q - I$ equitably 2-colored. This means that every graph $Q \in \mathcal{Q}_3(n)$ having an independent set $I$ of size $4n/10$ has coloring of type $(4n/10, 3n/10, 3n/10)$. Hence, we have

**Theorem 4.** *If $Q \in \mathcal{Q}_3(n)$ has an independent set $I$ of size $|I| \geq 4n/10$ then it has a semi-equitable coloring of type $(|I|, \lceil (n - |I|)/2 \rceil, \lfloor (n - |I|)/2 \rfloor)$.*

## 3 Example

Let us consider the following batch processing problem. Given three uniform processors $P_1, P_2, P_3$ such that one of them, say $P_1$ is twice slower than the remaining two, which are of the same speed. Next, we have 10 identical jobs to perform. Among the jobs there are incompatibility constrains such that each job is in conflict with exactly 3 other jobs, which means that they cannot be processed on the same processor. Our aim is to find a schedule that minimizes the makespan, i.e. maximal batch completion time, in symbols $Q3|s - batch, UET, G = cubic|C_{\max}$. Suppose that our jobs induce a cubic graph of the shape shown in Fig. 4. An optimal solution to this particular situation involves a semi-equitable coloring of $G$ and is shown in Fig. 5a. Fig. 5b depicts a suboptimal solution. Note that $C^*_{\max} = 4$ in this particular case. Also note that if $P_1$ is twice slower than the remaining two processors then the optimal value of schedule length fulfills

$$n/2.5 \le C^*_{\max} \le n/1.5. \tag{2}$$

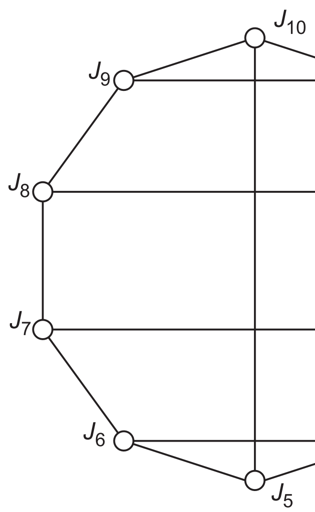


Figure 4: A scheduling graph $G$.

Let us notice that both bounds in (2) are tight. An example of achievement the lower bound is given in our example (Fig. 5a), while the proof for the tightness of upper bound follows from the case where the mutual exclusion graph is the prism (Fig. 2).

If processor $P_1$ is at least twice faster than the remaining two and the scheduling graph $G$ is bipartite then obtaining an optimal solution is easy. All we need is finding a 2-coloring of $G$ and splitting the second color into two thus obtaining a 3-coloring of type $(n/2, \lceil n/4 \rceil, \lfloor n/4 \rfloor)$. Then the vertices of the first color go to $P_1$ and the remaining vertices go to $P_2$ and $P_3$. If $P_1$ is faster than $P_2$ and $P_3$ but its speed factor is less than 2, we have to use a modified Chen's et. al. algorithm [1] which finds a

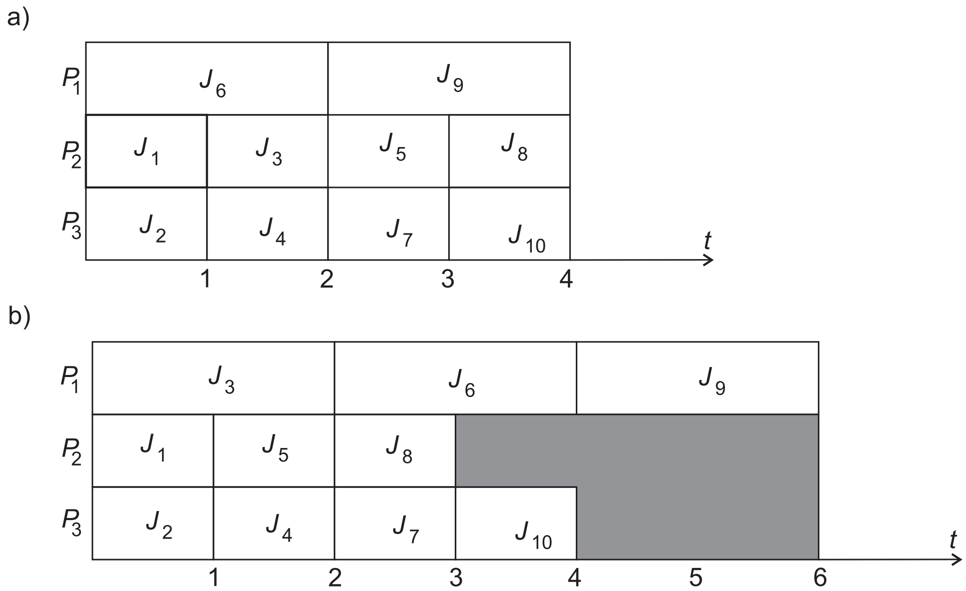


Figure 5: a) optimal schedule; b) suboptimal schedule.

required decomposition of a $G$ in $O(n^2)$ time. If, however, graph $G$ is tripartite then the scheduling problem becomes NP-hard.

## 4 Summary

Above, we have shown that every cubic graph with $k$ independent vertices has equitable coloring for $k \in \{\lceil n/3 \rceil, \lfloor n/3 \rfloor\}$ and semi-equitable coloring for $k \geq 4n/10$. The problem for $\lceil n/3 \rceil < k < 4n/10$ stays open. Our results on the guarantee of the existence of appropriate colorings in bicubic and tricubic graphs are summarized in Table 1.

Finally, note that our considerations cannot be generalized to all 3-colorable graphs, since the sun $S_3$ graph, depicted in Fig. 6, is a counterexample. There is an independent set $I$ of size 3 in $S_3$ such that $\chi(S_3 - I) = 3$, where $3 = n/2 > 4n/10$. Thus, the semi-equitable 3-coloring does not exist.

Table 1: The existence of appropriate 3-colorings in a function of size $I$, $|I| = k$.

| $\chi(Q)$ | 3-coloring | $k \in \{\lfloor n/3 \rfloor, \lceil n/3 \rceil\}$ | $\lceil n/3 \rceil < k < n/2.5$ | $n/2.5 \leq k < n/2$ |
|---|---|---|---|---|
| 2 | equitable | yes | yes | yes |
| | semi-equitable | no | yes | yes |
| 3 | equitable | yes | yes | yes |
| | semi-equitable | no | ? | yes |

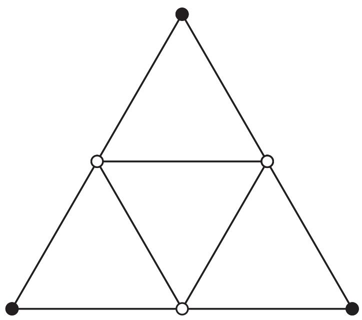

Figure 6: The sun graph $S_3$. The vertices in black indicate the independent set $I$.

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