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Asynchronous Sampling of a 2D Continuous Time Stochastic Differential Equation[1]

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Contents

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1 Introduction

In this report we present explicit formulae for the discrete-time prediction of the 2-dimensional process described by a continuous-time stochastic model. A standard procedure of prediction would require a numerical evaluation of the fundamental matrix e^{tA} and numerical integration of the object differential equation. We can, however, derive explicit equations for the state prediction in a 2-dimensional case (i.e. for state vectors with two coordinates) based on an explicit formula for calculating the matrix e^{tA}. Such a 2-dimensional model is useful in object tracking and robotics, for example, to describe the state of an object moving in the two dimensional Euclidean plane (with states corresponding to positions in X and Y). Of importance are also kinematic models with states corresponding to both position and velocity. The presented prediction method based on explicit equations is not numerically expensive and can be implemented on resource-constrained computers like embedded systems.

2 C-T Gauss-Markov Model

We consider the following 2-dimensional linear stochastic differential equation [2, 4]

$$\begin{aligned} dX(t) &= [AX(t) + b]\, dt + \sigma dw(t), \quad 0 \le t < \infty, \\ X(0) &= X_0, \end{aligned} \tag{1}$$

where w is the 2-dimensional Brownian motion independent of the initial vector X_0, which has a given 2-variate normal distribution. The (2×2), (2×1) and (2×2) matrices A, b, and σ, respectively, are nonrandom and bounded (and measurable [2]).

The solution of (1) has the following representation [2]:

$$\begin{aligned} X(t) &\triangleq \Phi(t) \left[X(0) + \int_0^t \Phi^{-1}(\eta) b d\eta + \int_0^t \Phi^{-1}(\eta) \sigma dw(\eta) \right], \\ &0 \le t < \infty, \end{aligned} \tag{2}$$

where $\Phi(t)$ is a nonsingular matrix called the *fundamental solution* to the following homogeneous ordinary differential equation

$$\dot{\zeta}(t) = A\zeta(t). \tag{3}$$

The fundamental solution to (3) can be calculated as

$$\Phi(t) = e^{tA} \triangleq \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i. \tag{4}$$

With the *mean vector* and *covariance matrix* functions defined as

$$\begin{aligned} m(t) &\triangleq EX(t), \\ \rho(t, \tau) &\triangleq E\left\{ [X(t) - m(t)] \left[X(\tau) - m(\tau) \right]^T \right\}, \\ V(t) &\triangleq \rho(t, t), \end{aligned} \tag{5}$$

[2] i.e., given a probability space $(\Omega, \mathcal{F}, P)$, and a measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, a function $f : \Omega \to \mathbb{R}$ is measurable, if for any Borel set $B \in \mathcal{B}(\mathbb{R})$, the inverse image $f^{-1}(B) \in \mathcal{F}$

it can be shown that

$$
\begin{aligned}
m(t) &= \Phi(t)\left[m(0) + \int_0^t \Phi^{-1}(\eta) b d\eta\right], \\
\rho(t,\tau) &= \Phi(t)\left[V(0) + \int_0^{t\cap\tau} \Phi^{-1}(\eta)\sigma \left[\Phi^{-1}(\eta)\sigma\right]^T d\eta\right]\Phi^T(\tau), \\
V(t) &= \Phi(t)\left[V(0) + \int_0^{t} \Phi^{-1}(\eta)\sigma \left[\Phi^{-1}(\eta)\sigma\right]^T d\eta\right]\Phi^T(t)
\end{aligned} \tag{6}
$$

hold for every $0 \leq t, \tau < \infty$, where $t \cap \tau \triangleq \min\{t, \tau\}$. It can be proved that X in (2) is a Gaussian process, thus the finite-dimensional distributions of the process X are completely determined by the mean and the covariance functions [2].

3 A Sampled-Data Model

For any two concrete time moments τ and t, $\tau \leq t$, the following transitional (reference) equation results from the c-t model described above by (1)–(6):

$$x(t) = F(t,\tau)x(\tau) + u(t,\tau) + w(t,\tau), \tag{7}$$

with

$$F(t,\tau) = \Phi(t)\Phi^{-1}(\tau), \tag{8}$$

$$u(t,\tau) = \Phi(t)\int_\tau^t \Phi^{-1}(\eta) b d\eta, \tag{9}$$

$$w(t,\tau) = \Phi(t)\int_\tau^t \Phi^{-1}(\eta)\sigma dw(\eta), \tag{10}$$

where (10) represents the Ito stochastic integral, which can be calculated by parts [3]. The mean of (10) is $E\{w(t,\tau)\} = 0, \forall t, \tau$, and the covariance matrix of (10) is

$$
\begin{aligned}
Q(t,\tau) &\triangleq E\left\{w(t,\tau)\left[w(t,\tau)\right]^T\right\} \\
&= \Phi(t)\left[\int_\tau^t \Phi^{-1}(\eta)\sigma\left[\Phi^{-1}(\eta)\sigma\right]^T d\eta\right]\Phi^T(t).
\end{aligned} \tag{11}
$$

A necessary discrete-time signal ($z \in \mathbb{R}^p$) of observations of the system $x(t)$, associated with the sensor can be described by the following equation:

$$z(t) = H(t)x(t) + \xi(t), \tag{12}$$

where $p \in \mathbb{N}_+$, $H(t)$ is a $p \times 2$ observation matrix, and $\xi \in \mathbb{R}^p$ represents zero-mean Gaussian discrete-time measurement noise with a known covariance matrix

$$R(t) \triangleq E\left\{\xi(t)\left[\xi(t)\right]^T\right\}. \tag{13}$$

4 Explicit Form of the Sample-Data Model

Repetitive numerical evaluation of (8), (9) and (11), necessary for the sample-data model (7), can be computationally expensive, thus their explicit forms are desirable. Therefore, in the following we show explicit equations for calculating the matrix e^A for the 2-dimensional case. Based on these results, explicit equations for calculating (8), (9) and (11) are then presented.

4.1 Calculating e^A

Let us consider a general real matrix A of size (2×2)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{14}$$

To calculate the matrix e^A, explicitly, three cases are considered [1]:

i) if $(a-d)^2 + 4bc = 0$, then

$$e^A = e^\alpha \left(I + \bar{A}\right), \tag{15}$$

ii) if $(a-d)^2 + 4bc > 0$, then

$$e^A = e^\alpha \left(\cosh(\delta) I + \frac{\sinh(\delta)}{\delta} \bar{A}\right), \tag{16}$$

iii) if $(a-d)^2 + 4bc < 0$, then

$$e^A = e^\alpha \left(\cos(\delta) I + \frac{\sin(\delta)}{\delta} \bar{A}\right), \tag{17}$$

where $\alpha = \frac{a+d}{2}$, I is the (2×2) identity matrix, $\bar{A}$ is defined as:

$$\bar{A} = \begin{bmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{bmatrix}, \tag{18}$$

and

$$\delta = \frac{\sqrt{|(a-d)^2 + 4bc|}}{2}. \tag{19}$$

4.2 Calculating F, u and Q

In the following section we use the same definitions of δ, I and $\bar{A}$ as in the previous section. Based on the above findings for the explicit calculation of the matrix e^A, the transition matrix (8), for any two time moments t and τ, $t > \tau$, can by calculated, for the three cases considered, as follows:

i)

$$\begin{aligned} F(t,\tau) &= e^{(t-\tau)A} = e^{\Delta A} \\ &= \exp\left\{\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right\} = \exp\left\{\begin{bmatrix} \Delta a & \Delta b \\ \Delta b & \Delta c \end{bmatrix}\right\} \\ &= e^{\Delta\alpha}\left(I + \Delta\bar{A}\right), \end{aligned} \tag{20}$$

and similarly

ii)

$$F(t,\tau) = e^{\Delta\alpha}\left(\cosh(\Delta\delta)I + \frac{\sinh(\Delta\delta)}{\delta}\bar{A}\right), \tag{21}$$

iii)

$$F(t,\tau) = e^{\Delta\alpha}\left(\cos(\Delta\delta)I + \frac{\sin(\Delta\delta)}{\delta}\bar{A}\right), \tag{22}$$

where $\Delta = t - \tau$, $\Delta > 0$.

Using the above results and the integral calculus, the discretized input (9) results in:

i)

$$u(t,\tau) = \left\{\Delta I + \frac{\Delta^2}{2}\bar{A}\right\}b, \qquad \text{for } \alpha = 0, \tag{23}$$

and

$$\begin{aligned} u(t,\tau) &= \left\{\left(\frac{e^{\Delta\alpha}}{\alpha}I - \frac{1}{\alpha}I\right) + \left(\frac{e^{\Delta\alpha}}{\alpha}(\Delta - \frac{1}{\alpha})\bar{A} + \frac{1}{\alpha^2}\bar{A}\right)\right\}b \\ &= \frac{1}{\alpha}\left\{e^{\Delta\alpha}\left[I + \left(\Delta - \frac{1}{\alpha}\right)\bar{A}\right] + \frac{1}{\alpha}\bar{A} - I\right\}b, \qquad \text{for } \alpha \neq 0 \end{aligned} \tag{24}$$

ii)

$$u(t,\tau) = \left\{\frac{\beta_1}{2}\left(I + \frac{1}{\delta}\bar{A}\right) + \frac{\beta_2}{2}\left(I - \frac{1}{\delta}\bar{A}\right)\right\}b, \tag{25}$$

where

$$\begin{aligned} \beta_1 &= \begin{cases} \left(e^{\Delta(\alpha+\delta)} - 1\right)/(\alpha+\delta), & \text{for } \alpha + \delta \neq 0, \text{ and,} \\ \Delta, & \text{for } \alpha + \delta = 0, \end{cases} \\ \beta_2 &= \begin{cases} \left(e^{\Delta(\alpha-\delta)} - 1\right)/(\alpha-\delta), & \text{for } \alpha - \delta \neq 0, \text{ and,} \\ \Delta, & \text{for } \alpha - \delta = 0. \end{cases} \end{aligned} \tag{26}$$

iii)

$$\begin{aligned} u(t,\tau) &= \left\{\left[e^{\Delta\alpha}\frac{\alpha\cos(\Delta\delta) + \delta\sin(\Delta\delta)}{\alpha^2+\delta^2} - \frac{\alpha}{\alpha^2+\delta^2}\right]I\right. \\ &\quad \left. + \left[e^{\Delta\alpha}\frac{\alpha\sin(\Delta\delta) - \delta\cos(\Delta\delta)}{\alpha^2+\delta^2} + \frac{\delta}{\alpha^2+\delta^2}\right]\frac{1}{\delta}\bar{A}\right\}b \\ &= \left\{\frac{e^{\Delta\alpha}\left[\cos(\Delta\delta)(\alpha I - \bar{A}) + \sin(\Delta\delta)(\delta I + (\alpha/\delta)\bar{A})\right] - \alpha I + \bar{A}}{\alpha^2+\delta^2}\right\}b. \end{aligned} \tag{27}$$

Though we assume the matrix b to be constant in time, it can be different for each integration interval $[\tau, t]$. Eventually, if the matrix $b = b(t)$ is a known function of time, the appropriate equations for such a case, equivalent to (23)–(27), can be derived.

Finally, the equations for the covariance matrix (11), for the three cases, are:

i)

$$Q(t,\tau) = \left(\bar{A}\sigma\sigma^T\bar{A}^T\right)\frac{\Delta^3}{3} + \left(\bar{A}\sigma\sigma^T + \sigma\sigma^T\bar{A}^T\right)\frac{\Delta^2}{2} + \sigma\sigma^T\Delta, \tag{28}$$

for $\alpha = 0$, and

$$\begin{aligned} Q(t,\tau) = \frac{1}{2\alpha}\Bigg[&\left(\bar{A}\sigma\sigma^T\bar{A}^T\right)\left(e^{2\Delta\alpha}\left(\Delta^2 - \frac{\Delta}{\alpha} + \frac{1}{2\alpha^2}\right) - \frac{1}{2\alpha^2}\right) \\ &+ \left(\bar{A}\sigma\sigma^T + \sigma\sigma^T\bar{A}^T\right)\left(\frac{1}{2\alpha} + e^{2\Delta\alpha}\left(\Delta - \frac{1}{2\alpha}\right)\right) \\ &+ \sigma\sigma^T\left(e^{2\Delta\alpha} - 1\right)\Bigg], \end{aligned} \tag{29}$$

for $\alpha \neq 0$.

ii)

$$\begin{aligned} Q(t,\tau) = &\frac{\beta}{2}\left(\sigma\sigma^T - \frac{\bar{A}\sigma\sigma^T\bar{A}^T}{\delta^2}\right) \\ &+ \frac{\beta_3}{4}\left(\sigma\sigma^T + \frac{\bar{A}\sigma\sigma^T\bar{A}^T}{\delta^2} + \frac{\bar{A}\sigma\sigma^T + \sigma\sigma^T\bar{A}^T}{\delta}\right) \\ &+ \frac{\beta_4}{4}\left(\sigma\sigma^T + \frac{\bar{A}\sigma\sigma^T\bar{A}^T}{\delta^2} - \frac{\bar{A}\sigma\sigma^T + \sigma\sigma^T\bar{A}^T}{\delta}\right), \end{aligned} \tag{30}$$

where

$$\beta = \begin{cases} (e^{2\Delta\alpha} - 1)/(2\alpha), & \text{for } \alpha \neq 0, \text{ and,} \\ \Delta, & \text{for } \alpha = 0, \end{cases} \tag{31}$$

and

$$\begin{aligned} \beta_3 &= \begin{cases} \left(e^{2\Delta(\alpha+\delta)} - 1\right)/\left(2(\alpha+\delta)\right), & \text{for } \alpha + \delta \neq 0, \text{ and,} \\ \Delta, & \text{for } \alpha + \delta = 0, \end{cases} \\ \beta_4 &= \begin{cases} \left(e^{2\Delta(\alpha-\delta)} - 1\right)/\left(2(\alpha-\delta)\right), & \text{for } \alpha - \delta \neq 0, \text{ and,} \\ \Delta, & \text{for } \alpha - \delta = 0. \end{cases} \end{aligned} \tag{32}$$

iii)

$$\begin{aligned} Q(t,\tau) = &\frac{\beta}{2}\left(\sigma\sigma^T + \frac{\bar{A}\sigma\sigma^T\bar{A}^T}{\delta^2}\right) + \\ &+ \frac{e^{2\Delta\alpha}\left[\alpha\cos(2\Delta\delta) + \delta\sin(2\Delta\delta)\right] - \alpha}{4(\alpha^2 + \delta^2)}\left(\sigma\sigma^T - \frac{\bar{A}\sigma\sigma^T\bar{A}^T}{\delta^2}\right) \\ &+ \frac{e^{2\Delta\alpha}\left[\alpha\sin(2\Delta\delta) - \delta\cos(2\Delta\delta)\right] + \delta}{4(\alpha^2 + \delta^2)}\left(\frac{\bar{A}\sigma\sigma^T + \sigma\sigma^T\bar{A}^T}{\delta}\right), \end{aligned} \tag{33}$$

with β defined as in (31).

5 Conclusions

In this report we have presented a method for discrete-time prediction of 2-dimensional object models. This method can be easily extended for higher dimensional models whenever the explicit formulae for the requisitive matrix exponential are available.

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