FUNDAMENTAL FREQUENCY SMOOTHING FOR NONSTATIONARY MULTI-HARMONIC SIGNALS

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ABSTRACT
A new estimator of the instantaneous fundamental frequency of a signal with multiple harmonic components is described. The proposed algorithm is a three-step procedure. First, the fundamental frequency is estimated using a bank of frequency tracking algorithms set for different signal harmonics. Then the smoothed estimates are evaluated by means of backward-time filtering of the results of tracking. Finally, the estimate of the instantaneous fundamental frequency is computed as a weighted sum of partial estimates. It is shown that the smoothing and weighting steps significantly increase both accuracy and robustness of frequency estimation.

1. INTRODUCTION
Consider the problem of estimation of the instantaneous fundamental frequency \( \omega_0(t) \) of the noisy complex-valued signal made up of \( K \) harmonic components \( s_k(t), k = 1, \ldots, K \) buried in wideband noise

\[
y(t) = \sum_{k=1}^{K} s_k(t) + v(t)
\]

where \( t = 0, 1, \ldots \) denotes discrete time, \( a_k(t) \) and \( \omega_k(t) \) denote the slowly time-varying instantaneous ‘amplitude’ (complex-valued – in order to incorporate the initial phase shift) and instantaneous frequency of the \( k \)-th signal component, and \( v(t) \) is a zero-mean circular white noise. We will assume that the frequencies are harmonically related, namely

\[
\omega_k(t) = k_0(t), \quad k = 1, \ldots, K
\]

where \( \omega_0(t) \) denotes the fundamental frequency of the signal.

In many applications where accurate estimation of the fundamental frequency of the signal is needed, one has an access to the ‘future’ signal samples, either due to the fact that the signal is pre-recorded (off-line applications), or because a certain processing delay is allowable (nearly real-time applications). In all such cases noncausal estimation, often called smoothing, offers substantial advantages over causal estimation (filtration) schemes. This occurs especially when the estimation problem is nonlinear, which is true in the case of frequency estimation [1], [2].

The aim of this paper is to provide a reliable smoothed estimate of \( \omega_0(t) \) based on prerecorded data sequence \( Y(N) = \{y(1), \ldots, y(N)\} \). Important applications that admit such problem formulation include contactless measurement of rotational speed of combustion engines using acoustic sensors (for diagnostic purposes) [3], and cancelation of harmonic interferences corrupting archive audio signals [4].

We show how to build a computationally simple fundamental frequency smoother. Our solution is based on the recently developed scheme for extraction/cancellation of narrowband signals [5]. Briefly speaking, we run several frequency estimation algorithms, set for different harmonic components of the signal, and combine the obtained partial estimation results in a statistically meaningful way. The extended scheme offers improved performance and has better robustness properties than the currently available solutions.

2. FREQUENCY ESTIMATION PROCEDURE
The proposed estimation procedure consists of three steps (see Fig. 1). First, the so-called pilot filters are run separately for each of the harmonic components of the signal. Each pilot filter provides preliminary estimates of the instantaneous frequency of the harmonic component it is tracking. Then the quality of frequency estimates yielded by pilot filters is improved by means of smoothing. Finally, the fundamental frequency estimates, derived from each of the harmonics, are combined together using a time-varying weighting scheme.

The proposed multi-stage parallel estimation scheme has several advantages. First, it allows one to significantly increase the estimation accuracy. Second, the performance of each of the subestimators can be monitored and at any time a decision can be made to exclude the poorly performing ones from the data fusion process; later on, when such deficient subestimators recover, they can again be incorporated into the decision process. Finally,
based on the same principle, the proposed structure can be easily extended when additional harmonic components emerge.

2.1. Frequency tracking

The algorithm that will be used for frequency tracking is a modified version of the adaptive notch filter described in [5]. This prototype algorithm has the following form

\[
\begin{align*}
ε(t) &= y(t) - \sum_{k=1}^{K} \hat{s}_k(t|t-1) \\
\hat{s}_k(t|t) &= \hat{s}_k(t|t-1) + με(t) \\
g_k(t) &= \text{Im} \left[ \frac{ε(t)}{\hat{s}_k(t|t-1)} \right] \\
\hat{ω}_k(t|t) &= \hat{ω}_k(t|t-1) + γg_k(t) \\
\hat{ω}_k(t|t+1) &= e^{j[\hat{ω}_k(t|t)+\hat{ω}_k(t|t-1)]+γg_k(t)} \\
k &= 1, \ldots, K \\
t &= 1, \ldots, N
\end{align*}
\]  

where \(\hat{s}_k(t+1|t)\) and \(\hat{s}_k(t|t)\) denote the predicted and filtered estimates of the \(k\)-th signal component, respectively, \(\hat{ω}_k(t)\) denotes the estimate of the \(k\)-th harmonic \(ω_k(t)\), and \(\hat{ω}_k(t|t)\) denotes the estimate of the rate of change of the \(k\)-th harmonic \(α_k(t) = ω_k(t) - ω_k(t-1)\). Finally, \(μ > 0, γ > 0\) and \(η > 0\), such that \(η ≪ γ ≪ μ\), denote user-dependent adaptation gains that control the rate of amplitude adaptation, frequency adaptation and frequency rate adaptation, respectively – for more details see [5].

The algorithm (3) is made up of \(K\) single-frequency adaptive notch filtering (ANF) subalgorithms, each taking care of one signal component, that work in parallel and are driven by the common prediction error \(ε(t)\).

The main drawback of the algorithm (3), when applied to solve the problem stated in this paper, is due to the fact that it was designed to extract/eliminate sinusoidal signal components with unrelated frequencies and hence it can’t exploit in any way the harmonic structure of the analyzed signal. Note that, according to (2), the estimates of the \(k\)-th harmonic \(ω_k(t)\) and of the corresponding rate of change \(α_k(t)\) can be used to generate the estimates of all other harmonics using the relationships

\[
\begin{align*}
\hat{w}_n|k(t|t) &= \frac{\hat{w}_n|k(t|t)η_n}{k}, \quad \hat{α}_n|k(t|t) = \frac{\hat{α}_k(t|t)η_n}{k} \\
n &= 1, \ldots, K, \quad n ≠ k
\end{align*}
\]  

This simple observation is the cornerstone of the new tracking algorithm summarized below

\[
\begin{align*}
ε_k(t) &= y(t) - \sum_{n=1}^{K} \hat{s}_n|k(t|t-1) \\
\hat{s}_n|k(t|t) &= \hat{s}_n|k(t|t-1) + με_k(t) \\
g_k(t) &= \text{Im} \left[ \frac{ε_k(t)}{\hat{s}_n|k(t|t-1)} \right] \\
\hat{ω}_k(t) &= \hat{ω}_k(t-1|t-1) + γg_k(t) \\
\hat{ω}_k(t|t+1) &= e^{j[\hat{ω}_k(t|t)+\hat{ω}_k(t|t-1)]+γg_k(t)} \\
\hat{s}_n|k(t+1|t) &= e^{j[\hat{ω}_k(t+1|t)+\hat{ω}_k(t|t-1)]} \hat{s}_n|k(t|t) \\
n &= 1, \ldots, K \\
\hat{s}_n|k(t+1|t) &= e^{j[\hat{ω}_k(t+1|t)+\hat{ω}_k(t|t-1)]} \hat{s}_n|k(t|t) \\
t &= 1, \ldots, N
\end{align*}
\]

Similar to (3), the new algorithm is composed of \(K\) subalgorithms, further referred to as pilot filters, which track different harmonics. However, unlike (3), the \(k\)-th pilot filter estimates on its own all signal components and works out its own prediction error \(ε_k(t)\). To avoid confusion, the corresponding estimates are double indexed – \(\hat{s}_n|k(t+1|t)\) and \(\hat{s}_n|k(t+1|t)\) denote the estimates of \(s_n(t)\) evaluated by the \(k\)-th pilot filter using (4). Such an estimation redundancy makes the constrained algorithm (5) more robust than the unconstrained one (3). When one of the subalgorithms in (3) fails to correctly track the signal component it is locked on (e.g. due to the temporary decrease of the corresponding signal-to-noise ratio), it adversely affects – via the common prediction error – all other subalgorithms. In extreme cases such a failure may even cause filter divergence. In the constrained algorithm chances for this to happen are much smaller.

Fig. 1. Block diagram of the proposed fundamental frequency smoother.
2.2. Frequency smoothing

For $K = 1$ the tracking analysis of the pilot filter was performed in [5]. We have shown there that the accuracy of the estimates can be further improved using a two-step linear noncausal filtration process. Here we extrapolate this idea to a more general case of $K > 1$. Even though we cannot support the proposed procedure with a rigorous analysis similar to that provided in [5], simulation results confirm its usefulness.

First, an intermediate sequence $\hat{\omega}_k(t), k \in \{1, \ldots, K\}$, is formed using the filter described by

$$\hat{\omega}_k(t) = \frac{bq^{-1}}{1 + cq^{-1}} \tilde{\omega}_k(t|t), \quad t = 1, \ldots, N$$

(6)

where $q^{-1}$ is the backward-shift operator, $q^{-1}\tilde{\omega}_k(t|t) = \tilde{\omega}_k(t-1|t-1)$, $b = \gamma/\eta$, and $c = -(\eta - \gamma)/\eta$.

Then the smoothed frequency estimates are formed from the intermediate sequence using the following purely anticausal filter

$$\tilde{\omega}_k(t) = \frac{\eta q}{1 + d_1 q + d_2 q^2 + d_3 q^3} \hat{\omega}_k(t)$$

(7)

$t = N, \ldots, 1$

where $d_1 = \mu + \eta + \gamma - 3$, $d_2 = 3 - 2\mu - \eta$, and $d_3 = \mu - 1$.

To obtain smoothed estimates of the fundamental frequency, one should perform simple scaling

$$\tilde{\omega}_{0|k}(t) = \tilde{\omega}_{1|k}(t) = \frac{\tilde{\omega}_k(t)}{k}, \quad k = 1, \ldots, K.$$  
(8)

2.3. Data fusion

For any harmonic $\omega_k(t)$, the mean-squared frequency estimation error can be written down as a sum of its bias component and variance component. In tracking analysis the bias error — the squared difference between the true frequency and the mean value of its estimate — is often called lag error, as it originates from the fact that the estimated frequency trajectory lags behind the true trajectory. The variance error, i.e., the mean-squared difference between the true frequency and the mean value of its estimate, is caused by fluctuations of frequency estimates due to measurement noise.

As shown in [5], when the instantaneous frequency varies slowly with time, smoothing practically removes the estimation bias, which means that $\mathbb{E}[\tilde{\omega}_0(t)] \cong \omega_0(t)$ and consequently

$$\mathbb{E}[\tilde{\omega}_{0|k}(t)] \cong \omega_0(t).$$

(9)

Importantly, since the smoothing filter (6) - (7) is ‘matched’ to the pilot algorithm, this result holds true for a wide range of adaptation gains $\mu$, $\gamma$ and $\eta$.

Using the analytical results presented in [5], one can also show that the variance error is inversely proportional to the signal-to-noise ratio (SNR)

$$\operatorname{var}[\tilde{\omega}_k(t)] \propto \frac{1}{\operatorname{SNR}_k(t)}, \quad \operatorname{SNR}_k(t) = \frac{\sigma^2_k(t)}{\sigma^2_j}.$$

This leads to

$$\operatorname{var}[\tilde{\omega}_{0|k}(t)] = \frac{\operatorname{var}[\tilde{\omega}_k(t)]}{k^2} \propto \frac{\sigma^2_j}{\sigma^2_k(t)}k^2.$$  
(10)

It is well known [6] that if $X_1, \ldots, X_K$ are independent random variables with mean $\mu$ and known variances $\sigma^2_1, \ldots, \sigma^2_K$, the optimal (in the mean-square sense) linear estimator of $\mu$ takes the form of a weighted average

$$\tilde{\mu} = \sum_{k=1}^K \lambda_k X_k,$$

where the weights $\lambda_k$ are inversely proportional to the variances of the corresponding random variables

$$\lambda_k = \frac{1/\sigma^2_k}{\sum_{k=1}^K 1/\sigma^2_k}.$$  
(11)

Based on (9) and (10), we can form the final estimate of the fundamental frequency in an analogous fashion

$$\tilde{\omega}_0(t) = \sum_{k=1}^K \lambda_k(\tilde{\omega}_{0|k}(t)).$$

(12)

where $\lambda_k(t), k = 1, \ldots, K$, are time-dependent weights. Since the exact variances of $\tilde{\omega}_{0|k}(t)$ are not known, the weights must be evaluated approximately. In agreement with (10) and (11), we propose to compute the weights using the following formula

$$\lambda_k(t) = \frac{1}{\sum_{i=1}^K |\tilde{s}_{i|k}(t)|^2}.$$  
(13)

Observe that in the proposed scheme ‘strong’ harmonics are assigned larger weights than the ‘weak’ ones. On the qualitative level such weighting seems to be reasonable since the strong signal components, i.e., those characterized by large SNR values, are tracked more accurately than the weak components. Simulation experiments, presented in the next section, confirm that the scheme (13) provides weights that remain very close to the optimal ones.

3. ESTIMATION OF THE RATE OF FREQUENCY CHANGE

Following [5], the smoothed estimates of the rates of frequency change $\alpha_k(t), k = 1, \ldots, K$, can be obtained by means of backward-time filtering of the estimates yielded by the corresponding pilot filters

$$\tilde{\alpha}_k(t) = \frac{\eta q}{1 + d_1 q + d_2 q^2 + d_3 q^3} \tilde{\alpha}_k(t|t).$$

(14)

$t = N, \ldots, 1$

where the filter coefficients $d_1, d_2$, and $d_3$ are identical with those used in (7).

The smoothed estimates of the rate of change of the fundamental frequency can be obtained from

$$\tilde{\alpha}_{0|k}(t) = \tilde{\alpha}_{1|k}(t) = \frac{\tilde{\alpha}_k(t)}{k}, \quad k = 1, \ldots, K.$$  
(15)
Finally, the combined (weighted) estimate takes the following form, analogous to (12)

$$\tilde{\alpha}_0(t) = \sum_{k=1}^{K} \lambda_k(t) \tilde{\alpha}_{0|k}(t).$$

(16)

### 4. SIMULATION RESULTS

The tracking accuracy of the proposed algorithm was verified using simulations. An artificial signal, consisting of three harmonics and white measurement noise was generated. The instantaneous fundamental frequency was varying in the sinusoidal fashion between 0.15 and 0.45 rad/Sa (Sa denotes the ‘sample’ unit) with a period of 2000 samples (Fig. 2). The variance of the measurement noise was equal to 0.02. The amplitudes of the three harmonic components were varying as depicted in Fig. 3. Note that the signal-to-noise ratio varied greatly for each of the harmonics, and that location of the strongest component changed several times during the experiment.

All pilot filters shared the same values of adaptation gains: $\mu = 0.05$, $\eta = \mu^2/2$, $\gamma = \mu \eta/4$, $k = 1, \ldots, 3$ (the values of frequency-tracking gains $\eta$ and $\gamma$ stem from the rule of thumb suggested in [5]).

To show how the subsequent processing steps improve estimation accuracy, we present both the results yielded by the proposed schemes and the intermediate results obtained for each of the component pilot filters and smoothers. Furthermore, since in the simulation experiment the true values of variances $\text{var}[^{\omega}_{0|k}(t)]$, $\text{var}[^{\tilde{\alpha}}_{0|k}(t)]$, $k = 1, \ldots, 3$, can be easily established by means of ensemble averaging, the proposed weighting scheme is also compared with the optimal one. Additionally, we compare its performance with that of simpler, winner-takes-all smoothers which at any time instant $t$ pick frequency and frequency rate estimates obtained for the strongest signal component

$$^{\omega}_0(t) = ^{\omega}_{0|k^*(t)}(t), \quad ^{\tilde{\alpha}}_0(t) = ^{\tilde{\alpha}}_{0|k^*(t)}(t)$$

where $k^*(t) = \arg \max_k |^{\tilde{s}}_{k,k}(t)|^2$.

The detailed summary of the steady state mean-squared errors, yielded by different estimation algorithms described in this paper, is shown in Table 1. The results were obtained by combined ensemble averaging [50 realizations of $\nu(t)$] and time averaging ($t \in [1000, 4000]$).

Note the gross improvement introduced by smoothing. The weighting scheme provides further noticeable error reduction. Observe also that the scheme based on the choice of the strongest harmonic performs worse than that based on adaptive weighting, and that the latter one nearly matches the scheme incorporating optimal weighting.

To check how the gains of pilot filters affect final results, additional simulations were performed with $\mu$ taking several values between 0.01 and 0.1. Fig. 4 depicts the results. Below 0.03 and above 0.07 the performance drops significantly. This negative effect, which was caused by some of the pilot filters locking to wrong harmonics, can be eliminated by introduction of a special supervisory layer capable of detecting such events.

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**Fig. 2.** The instantaneous fundamental frequency $\omega_0(t)$ of the signal used in the simulation experiment.

**Fig. 3.** Instantaneous amplitudes of the three harmonic components in the simulated signal: 1-st harmonic (solid line), 2-nd harmonic (dashed line) and 3-rd harmonic (dotted line).

**Fig. 4.** Dependence of the steady state mean-squared fundamental frequency estimation error on the gain $\mu$. 

Table 1. Steady state mean-squared errors of the estimated fundamental frequency (MSE$_{\omega_0}$) and its rate of change (MSE$_{\omega_0'}$) for different algorithms.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>MSE$_{\omega_0}$</th>
<th>MSE$_{\omega_0'}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pilot, 1-st harmonic</td>
<td>1.70 · 10^{-9}</td>
<td>8.07 · 10^{-9}</td>
</tr>
<tr>
<td>Pilot, 2-nd harmonic</td>
<td>1.16 · 10^{-5}</td>
<td>7.15 · 10^{-9}</td>
</tr>
<tr>
<td>Pilot, 3-rd harmonic</td>
<td>1.10 · 10^{-5}</td>
<td>7.07 · 10^{-9}</td>
</tr>
<tr>
<td>Smoother, 1-st harmonic</td>
<td>2.34 · 10^{-7}</td>
<td>1.01 · 10^{-10}</td>
</tr>
<tr>
<td>Smoother, 2-nd harmonic</td>
<td>3.20 · 10^{-8}</td>
<td>1.01 · 10^{-11}</td>
</tr>
<tr>
<td>Smoother, 3-rd harmonic</td>
<td>1.86 · 10^{-8}</td>
<td>4.70 · 10^{-12}</td>
</tr>
<tr>
<td>Strongest component</td>
<td>2.29 · 10^{-8}</td>
<td>6.39 · 10^{-12}</td>
</tr>
<tr>
<td>Adaptive weighting</td>
<td>1.54 · 10^{-8}</td>
<td>3.24 · 10^{-12}</td>
</tr>
<tr>
<td>Optimal weighting</td>
<td>1.52 · 10^{-8}</td>
<td>3.08 · 10^{-12}</td>
</tr>
</tbody>
</table>

5. REAL-WORLD EXPERIMENT

The proposed algorithm was applied to a real-world acoustic signal - a sound of a motorcycle engine noise - sampled at a frequency of 1100 Hz. The four second long recording includes the periods of acceleration (twice), gear shift and braking. The spectrogram of the signal, which consists of $K = 12$ harmonics, is shown in Fig. 5.

The complex-valued version of the signal was obtained using the Hilbert transform. Similar results (slightly inferior to those presented below) can be obtained by processing the real-valued signal using the modified version of the algorithm (5). The only change needed is replacement of the complex-valued prediction error with its real-valued counterpart: $e_k(t) = y(t) - \sum_{n=1}^{K} \text{Re} \left[ s_n(t) \gamma \right] (t - 1)$.

Similar to our simulation experiment, the adaptation gains of all pilot trackers were set to $\mu = 0.05$, $\gamma_0 = \mu^2 / 2$ and $\gamma_0 = \mu \gamma_0 / 4$. Due to the high degree of signal nonstationarity and abrupt changes of the signal-to-noise ratio, the proposed algorithm was able to successfully track only the third, fifth and eighth harmonics. Tracking of other harmonics was not reliable - the algorithm either locked on a harmonic different from the desired one (the initial frequency estimates were based on analysis of the periodogram of the first 256 signal samples), or it switched to another harmonic in the gear shifting phase, where SNR was pretty low. We note that both problems indicated above can be solved by adding a special supervisory layer, monitoring performance of all pilot trackers and enforcing consistency among the obtained estimation results.

Figures 6 and 7 show the frequency estimates yielded by the pilot filters and smoothers, respectively. It is clear that in all three cases smoothing increased tracking accuracy considerably.

Fig. 8 shows the fundamental frequency estimates and the rate of frequency change estimates obtained using the proposed weighting scheme. Even though we do not have any means to confirm that the weighing procedure improves the estimation accuracy, we can make some qualitative observations that support such a claim. During both periods where the engine accelerates in a steady fashion, all three smoothers yield similar, highly accurate results. However, during the period where the gear shift occurs, one of the smoothed frequency trajectories differs significantly from the other two ones – see Fig. 9. Observe that in this case the weighed estimates follow a more consistent track. The algorithm continues to provide reliable results even when the SNR drops to its lowest value, just prior to the second acceleration episode.
Finally, it is instructive to look at Fig. 10, which shows comparison of the smoothed estimates of the rate of change $\alpha_0(t)$ computed according to (16), with those obtained by means of ‘differentiating’ the smoothed instantaneous frequency estimates

$$\tilde{\alpha}_0(t) = \tilde{\omega}_0(t) - \tilde{\omega}_0(t-1).$$

The advantages of direct estimation of $\alpha_0(t)$ are clearly visible.

6. CONCLUSION

A new fundamental frequency smoother for nonstationary multi-harmonic signals was described. The proposed estimation procedure consists of three steps. First, the so-called pilot filters are run separately for each of the harmonic components of the signal. Then the quality of frequency estimates yielded by pilot filters is improved by means of smoothing. Finally, the fundamental frequency estimates, derived from each of the harmonics, are combined together using a time-varying weighting scheme. The new algorithm offers improved performance and has better robustness properties than the currently available ones.

7. REFERENCES


