ON THE INSTANTANEOUS FREQUENCY SMOOTHING FOR SIGNALS WITH QUASI-LINEAR FREQUENCY CHANGES

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ABSTRACT
The problem of estimation of the slowly-varying instantaneous frequency of a nonstationary complex sinusoidal signal buried in noise is considered. This problem is usually solved using frequency tracking algorithms. It is shown that the accuracy of frequency estimates can be considerably increased if the results yielded by the frequency tracker are further processed using the appropriately designed filters. The resulting frequency smoother can be employed in many off-line applications. Whenever signal frequency varies in a sufficiently smooth manner, the proposed algorithm, based on a new, quasi-linear model of frequency changes, outperforms the existing solutions.

Index Terms— Adaptive signal processing, frequency smoothing.

1. INTRODUCTION
The need to estimate the parameters of nonstationary sinusoidal signals embedded in noise arises in many practical applications, such as radar and sonar processing, or biomedical inference – see e.g. [1] and the references therein. The majority of known contributions to the problem of time-varying frequency estimation focus on on-line frequency tracking, i.e., they describe recursive algorithms that allow one to update the frequency estimates when the new measurements become available. When frequency changes can be modeled as a random-walk process, at least several of the proposed frequency trackers can be shown (under Gaussian assumptions) to achieve the posterior Cramér-Rao-type lower tracking bound (LTB), which limits tracking accuracy of any causal estimation scheme [2]. Some algorithms are also available that achieve LTB [3], or nearly achieve it [4], when the instantaneous frequency obeys the second-order random-walk model.

Compared to frequency tracking, the problem of time-varying frequency smoothing has been much less explored. By smoothing we mean noncausal estimation procedures that provide estimates of the instantaneous frequency based on both “past” and “future” (prerecorded) measurements. When appropriately designed, such noncausal estimators yield smaller estimation errors than their causal counterparts. For this reason frequency smoothers constitute an attractive alternative to frequency trackers in all applications that do not require on-line processing.

For random-walk frequency changes the problem of frequency smoothing was studied in [5] and [6] – in both cases the proposed algorithms are statistically efficient, i.e., when optimally tuned they achieve the lower smoothing bound (LSB). In this paper we extend the results presented in [6] to signals with quasi-linearly modulated frequency. The new frequency smoother is statistically efficient (under idealistic assumptions), but also – which is much more important from the practical viewpoint – it is robust to frequency/amplitude model misspecification.

2. PROBLEM STATEMENT
Consider the problem of estimation of the slowly-varying instantaneous frequency \( \omega(t) \in (-\pi, \pi] \) of a nonstationary, complex-valued sinusoidal signal (cisoid) \( s(t) \), based on \( N \) prerecorded noisy measurements \( y(t) \):

\[
\begin{align*}
    s(t) &= af(t), \quad a = a_0e^{j\phi_0}, \quad f(t) = e^{j\sum_{l=1}^{t} \omega(l)} \\
    y(t) &= s(t) + v(t) 
\end{align*}
\]

where \( t = 1, \ldots, N \) denotes the normalized discrete time, \( a_0 \) is the real-valued signal amplitude, \( \phi_0 \) stands for the initial signal phase, and \( v(t) \) denotes white measurement noise obeying

(A1) \( \{v(t)\} \) is a zero-mean circular white sequence with variance \( \sigma_v^2 \).

2.1. Quasi-linear Frequency Changes
Consider frequency variations governed by the following model

\[
\omega(t + 1) = \omega(t) + \alpha(t), \quad \alpha(t) = \alpha(t - 1) + w(t) \tag{2}
\]

where \( \alpha(t) \) denotes frequency rate and \( w(t) \) denotes the one-step frequency rate change. According to (2) it holds that...
(1 − q−1)2ω(t) = q−1w(t) where q−1 is the backward shift operator. Since (1 − q−1)2ω(t) = 0 implies ω(t) = ω0 + δωt, (2) can be regarded as a perturbed linear growth/decay model. When it holds that

(A2) \{w(t)\}, independent of \{v(t)\}, is a zero-mean white sequence with variance \sigma_w^2,
equations (2) define the so-called second-order random-walk model. The corresponding frequency changes will be further referred to as quasi-linear.
To claim statistical efficiency we will also need the following assumption

(A3) The sequences \{v(t)\} and \{w(t)\} are normally distributed.

2.2. Pilot frequency tracker and its properties

Consider the following pilot algorithm, which combines frequency tracking with frequency rate tracking

\[ f(t) = e^{i(\omega(t-1) + \delta(t-1))f(t-1)} \]
\[ \varepsilon(t) = y(t) - \hat{a}(t-1)f(t) \]
\[ \delta(t) = \text{Im} \left[ \frac{\varepsilon(t)}{\hat{a}(t-1)f(t)} \right] \]
\[ \hat{a}(t) = \hat{a}(t-1) + \mu \hat{\varepsilon}(t) \]
\[ \hat{\omega}(t) = \hat{\omega}(t-1) + \hat{a}(t-1) + \gamma_\alpha \delta(t) \]

where \* denotes complex conjugation, and \mu > 0, \gamma_\omega > 0, \gamma_\alpha > 0, \gamma_\alpha \ll \gamma_\omega \ll \mu, are small adaptation gains determining the rate of amplitude adaptation, frequency adaptation and frequency rate adaptation, respectively.

We note that (3) is a special case of the algorithm proposed in [7] for identification of quasi-periodically varying systems. As shown there, when applied to signals governed by (1) obeying assumptions (A1)–(A3), and when optimally tuned, it is a statistically efficient frequency tracker.

For \gamma_\alpha = 0 and under zero initial conditions \hat{a}(0) = 0, (3) reduces down to the algorithm studied in [6] – the equivalence holds for \gamma = \gamma_\omega, where \gamma is the adaptation gain used in [6].

Tracking properties of the pilot algorithm can be analyzed using the approximating linear filter (ALF) technique – the stochastic linearization approach described in [5]. It can be shown that the frequency estimation error \Delta \hat{\omega}(t) = \omega(t) - \hat{\omega}(t) can be approximately expressed in the form

\[ \Delta \hat{\omega}(t) = H_1(q^{-1})e(t) + H_2(q^{-1})w(t) \]

where \[ e(t) = -\text{Im}\{v(t)s^*(t)/a_0^2\} \], is a zero-mean white noise, independent of \{w(t)\}, with variance \sigma_e^2 = \sigma_v^2/(2a_0^2), \nH_1(q^{-1}) = (1 - q^{-1})(\gamma_\omega + (\gamma_\alpha - \gamma_\omega)q^{-1})/D(q^{-1}) \]
\nH_2(q^{-1}) = q^{-1}[1 - \gamma_\omega - (1 - \mu)q^{-1})]/D(q^{-1}) \]
\nD(q^{-1}) = 1 + d_1q^{-1} + d_2q^{-2} + d_3q^{-3} \]
and \[ d_1 = \mu + \gamma_\omega + \gamma_\alpha - 3, d_2 = 3 - 2\mu - \gamma_\omega, d_3 = \mu - 1. \]

All filters are asymptotically stable if adaptation gains fulfill the following (sufficient) stability conditions: \[ 0 < \mu < 1, \]
\[ 0 < \gamma_\omega < 1, 0 < \gamma_\alpha < 1 \]
\[ \mu(\gamma_\omega + \gamma_\alpha) > \gamma_\alpha. \]

It is worth noting that ALF approximations remain valid for any uniformly small sequences \{v(t)\} and \{w(t)\}, i.e., they are not restricted to sequences obeying assumptions (A1)–(A3). Additionally, the functional form of ALF equations does not change when signal amplitude is also slowly varying with time. These facts have important implications when it comes to robustness analysis of the smoothing algorithm.

3. FREQUENCY SMOOTHING

The pilot algorithm (3) and its approximate error model (4) will serve as a starting point for derivation of an adaptive frequency smoother. Basically, we will follow the steps of [6], i.e., we will show that the smoothed frequency estimates can be obtained by means of postfiltering the estimated frequency trajectory yielded by the pilot tracker. Additionally, we will argue why the proposed approach to frequency smoothing should be robust to modeling errors.

3.1. Optimization

Suppose that the entire measurement record is available up to the instant \[ N: Y(N) = \{y(1), \ldots, y(N)\}. \]
To obtain a fixed-interval smoothed estimate of \omega(t), further denoted by \hat{\omega}(t), we will pass the estimates \hat{\omega}(t), \[ t = 1, \ldots, N \]
through a noncausal filter \[ P(q^{-1}) = \ldots + p_{-1}q^{-1} + p_0 + p_1q + \ldots \]

\[ \hat{\omega}(t) = P(q^{-1})\hat{\omega}(t) \]

The filter \[ P(q^{-1}) \] will be designed so as to minimize the mean-squared frequency estimation error \[ E\{[\Delta \hat{\omega}(t)]^2\} \], where \[ \Delta \hat{\omega}(t) = \omega(t) - \hat{\omega}(t) \]. After elementary but tedious calculations, one arrives at

\[ \Delta \hat{\omega}(t) = (1 - q^{-1})Y(q^{-1})e(t) + \frac{1 - Y(q^{-1})}{(1 - q^{-1})^2} w(t-1) \]

where

\[ Y(q^{-1}) = P(q^{-1})Q(q^{-1}) \]
\[ Q(q^{-1}) = 1 - q(1 - q^{-1})H_2(q^{-1}) \]
\[ = [\gamma_\omega + (\gamma_\alpha - \gamma_\omega)q^{-1})]/D(q^{-1}) \]

Let \[ \Delta(q^{-1}) = 1/(1 - q^{-1}) \]. Due to orthogonality of \{e(t)\} and \{w(t)\}, one obtains

\[ E\{[\Delta \hat{\omega}(t)]^2\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g[Y(e^{-j\xi})] d\xi \]

where

\[ g[Y] = \frac{Y^*}{|\Delta|^2} \sigma_e^2 + |\Delta|^4(1 - Y)(1 - Y^*) \sigma_w^2 \]

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Minimization of (9) can be done by minimizing $g[Y(e^{-j\xi})]$ for every value of $\xi \in (-\pi, \pi]$. Using Wirtinger calculus [1], one arrives at the following formula for the optimal transfer function $Y_0(q^{-1})$

$$Y_0(q^{-1}) = \frac{2\kappa}{2\kappa + (1 - q^{-1})^2(1 - q)^2} = S_0(q^{-1})S_0(q)$$

$$S_0(q^{-1}) = \frac{\gamma_0q^{-1}}{D(q^{-1})}$$

(10)

where $\kappa = \alpha_2^2\sigma_w^2/\sigma_v^2$ denotes the rate of signal nonstationarity [6], and $\mu^{opt}$, $\gamma^{opt}$, $\gamma^{opt}_a$ denote the optimal settings for the pilot tracker.

After combining (7) with (10), one arrives at

$$P_0(q^{-1}) \cong \frac{S_0(q^{-1})S_0(q)}{Q_0(q^{-1})}$$

(11)

where $Q_0(q^{-1}) = Q(q^{-1})|_{\mu^{opt}, \gamma^{opt}_a}$. It can be shown [7] that under assumptions (A1)–(A3) the resulting optimized smoothing scheme attains the lower frequency smoothing bound, which means that for quasi-linear frequency changes it is statistically efficient.

### 3.2. Robust smoothing scheme

The optimality results are of little practical value – they are restricted to a specific model of frequency changes and they were derived under assumption that the optimal settings for the pilot tracker are known. Based on (11), we propose the following robust postfiltering scheme

$$\tilde{\omega}(t) = S(q)T(q^{-1})\tilde{\omega}(t)$$

(12)

where

$$T(q^{-1}) = \frac{S(q^{-1})}{Q(q^{-1})} = \frac{\gamma_0q^{-1}}{\gamma_0 + (\gamma_0 - \gamma_\omega)q^{-1}} = \frac{b_1q^{-1}}{1 + c_1q^{-1}}$$

and $b_1 = \gamma_0/\gamma_\omega$, $c_1 = (\gamma_0 - \gamma_\omega)/\gamma_\omega$.

Note that the filter $T(q^{-1})$ is causal and the filter $S(q)$ is anticausal. Therefore, postfiltering can be realized by means of forward-time filtering of the frequency trajectory $\{\tilde{\omega}(t)\}$ using the filter $T(q^{-1})$, followed by backward-time filtering of the results using the filter $S(q)$ - see Tab. 1.

Our robustness claim is based on the following relationship

$$\tilde{\omega}(t) = Q(q^{-1})\omega(t) - H_1(q^{-1})e(t)$$

(13)

which is an equivalent of (4).

Note that for the zero-mean measurement noise it holds that $E[e(t)] = 0$, leading to (in steady-state)

$$E[\tilde{\omega}(t)|\omega(s), 1 \leq s \leq N] = S(q)T(q^{-1})Q(q^{-1})\omega(t) = S(q)S(q^{-1})\omega(t) \cong \omega(t)$$

(14)

for any frequency trajectory that can be modeled as a lowpass signal. The second transition in (14) stems from the fact that $S(q^{-1})$ is a lowpass filter with the unity static gain $S(1) = 1$.

### Table 1. Frequency smoothing algorithm

<table>
<thead>
<tr>
<th>frequency tracker</th>
<th>$\hat{\omega}(1)$</th>
<th>$\hat{\omega}(t)$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\omega}(1)$</td>
<td>$\hat{\omega}(1)$</td>
<td>$\hat{\omega}(1)$</td>
<td>$t$</td>
</tr>
<tr>
<td>$\hat{\omega}(t)$</td>
<td>$-c_1\hat{\omega}(t - 1) + b_1\hat{\omega}(t - 1)$</td>
<td>$\hat{\omega}(t)$</td>
<td>$t = 2, \ldots, N$</td>
</tr>
<tr>
<td>$\hat{\omega}(t)$</td>
<td>$-d_1\hat{\omega}(t + 1) + d_2\hat{\omega}(t + 2)$</td>
<td>$\hat{\omega}(t)$</td>
<td>$t = N + 1, N - 2$</td>
</tr>
<tr>
<td>$\hat{\omega}(t)$</td>
<td>$-d_3\hat{\omega}(t + 3) + \gamma_\omega\hat{\omega}(t + 1)$</td>
<td>$\hat{\omega}(t)$</td>
<td>$t = N - 3, \ldots, 1$</td>
</tr>
</tbody>
</table>

According to (14), the smoothed frequency estimates are approximately unbiased for any choice of adaptation gains $\mu$, $\gamma_\omega$ and $\gamma_\alpha$ that guarantee stable operation of the pilot tracker. Importantly, since the ALF approximation (4), and hence also the relationship (13), remain valid for any uniformly small sequence of one-step frequency rate changes, this conclusion holds true irrespective of the shape of the estimated frequency trajectory (as long as it is sufficiently smooth). It also extends to cissoids with slowly-varying amplitudes.

### 4. SIMULATION RESULTS

Our simulation study will focus on two aspects of the proposed smoothing scheme: optimality and robustness. Although demonstration of the algorithm’s optimality, i.e., its ability to reach the Cramér-Rao-type lower smoothing bound is mainly of theoretical value, it is important as it allows one to specify conditions under which some absolute performance limits can be reached.

From the practical viewpoint, the most important property of the estimation algorithm is its robustness, i.e., insensitivity to modeling errors. We will show that, exactly as predicted, the proposed smoothing algorithm outperforms its tracking counterpart for a wide range of operating conditions, including different (nonstandard) amplitude/frequency trajectories and different (non-optimal) values of adaptation gains.
Fig. 1 shows comparison of theoretical values of the lower smoothing bounds with experimental results obtained for the signal (1) obeying assumptions (A1)–(A3), 3 different SNR values (0 dB, 10 dB and 20 dB) and 13 different values of the rate of nonstationarity $\kappa$, ranging from $10^{-10}$ to $10^{-4}$. There are no results for SNR=0 dB and $\kappa \geq 10^{-6}$ since the algorithm was unable to track under such extremely difficult conditions. All MSE values were computed for the optimally tuned algorithm by means of joint time and ensemble averaging. Note the excellent agreement between the theoretical curves and the results of computer simulations.

4.1. Optimality

4.2. Robustness

To check performance of the smoothing algorithm, a noisy quasiperiodically varying signal (2) was generated with the amplitude and frequency changes governed by $a(t) = 1 + 0.5 \cos(2\pi t/2000)$ and $\omega(t) = \sin(2\pi t/2000)$, respectively. Fig. 2 shows the comparison of the steady-state mean-squared frequency estimation errors, yielded by the tracking algorithm without frequency rate estimation, the smoothing algorithm without frequency rate estimation, the tracking algorithm with frequency rate estimation, and the smoothing algorithm with frequency rate estimation. The comparison was made for 40 different values of the adaptation gain $\mu$ and $\sigma_\alpha = 0.1$ (SNR=20 dB). To reduce the number of design degrees of freedom, the two other gains adopted for the algorithms with frequency rate estimation were set to: $\gamma_\omega = \mu^2/2$ and $\gamma_\alpha = \mu \gamma_\omega / 4$ (for justification see [7]). The algorithms without frequency rate estimation were obtained by setting $\gamma_\alpha = 0$. All MSE values were obtained by means of joint time averaging (the evaluation interval [2001,8000] was placed inside a wider analysis interval [1,10000]), and ensemble averaging (100 realizations of measurement noise were used). As expected, the smoothing algorithms yielded uniformly better results than their tracking counterparts. The achievable variance reduction is approximately equal to 20 dB.

5. REFERENCES


